

Notes on the biextension of Chow groups

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Abstract

The paper discusses four approaches to the biextension of Chow groups and their equivalences. These are the following: an explicit construction given by S. Bloch, a construction in terms of the Poincaré biextension of dual intermediate Jacobians, a construction in terms of K -cohomology, and a construction in terms of determinant of cohomology of coherent sheaves. A new approach to J. Franke's Chow categories is given. An explicit formula for the Weil pairing of algebraic cycles is obtained.

1 Introduction

One of the questions about algebraic cycles is the following: what can be associated in a bilinear way to a pair of algebraic cycles (Z, W) of codimensions p and q , respectively, on a smooth projective variety X of dimension d over a field k with $p + q = d + 1$, i.e., what is an analogue of the linking number for algebraic cycles? This question arose naturally from some of the approaches to the intersection index of arithmetic cycles on arithmetic schemes, i.e., to the height pairing for algebraic cycles.

For homologically trivial cycles, a “linking invariant” was constructed by S. Bloch in [5] and [7], and independently by A. Beilinson in [3]. It turns out that for an *arbitrary* ground field k , this invariant is no longer a number, but it is a k^* -torsor, i.e., a set with a free transitive action of the group k^* . More precisely, in [7] a biextension P of $(CH^p(X)_{\text{hom}}, CH^q(X)_{\text{hom}})$ by k^* is constructed, where $CH^p(X)_{\text{hom}}$ is the group of codimension p homologically trivial algebraic cycles on X up to rational equivalence. This means that there is a map of sets

$$\pi : P \rightarrow CH^p(X)_{\text{hom}} \times CH^q(X)_{\text{hom}},$$

a free action of k^* on the set P such that π induces a bijection

$$P/k^* \cong CH^p(X)_{\text{hom}} \times CH^q(X)_{\text{hom}},$$

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and for all elements $\alpha, \beta \in CH^p(X)_{\text{hom}}$, $\gamma, \delta \in CH^q(X)_{\text{hom}}$, there are fixed isomorphisms

$$P_{(\alpha, \gamma)} \otimes P_{(\beta, \gamma)} \cong P_{(\alpha + \beta, \gamma)}$$

$$P_{(\alpha, \gamma)} \otimes P_{(\alpha, \delta)} \cong P_{(\alpha, \gamma + \delta)}$$

such that certain compatibility axioms are satisfied (see [9]). Here the tensor product is taken in the category of k^* -torsors and $P_{(*, *)}$ denotes a fiber of P at $(*, *) \in CH^p(X)_{\text{hom}} \times CH^q(X)_{\text{hom}}$. In other words, the biextension P defines a bilinear pairing between Chow groups of homologically trivial cycles with value in the category of k^* -torsors.

The biextension P generalizes the Poincaré line bundle on the product of the Picard and Albanese varieties. Note that if the ground field k is a *number field*, then each embedding of k into its completion k_v induces a trivialization of the biextension $\log |P|_v$ and the collection of all these trivializations defines in a certain way the height pairing for algebraic cycles (see [5]).

On the other hand, the biextension P of $(CH^p(X)_{\text{hom}}, CH^q(X)_{\text{hom}})$ by k^* for $p + q = d + 1$ is an analogue of the intersection index $CH^p \times CH^q(X) \rightarrow \mathbb{Z}$ for $p + q = d$. There are several approaches to algebraic cycles, each of them giving its own definition of the intersection index. A natural question is to find analogous definitions for the biextension P . The main goal of the paper is to give a detailed answer to this question. Namely, we discuss four different constructions of biextensions of Chow groups and prove their equivalences.

Let us remind several approaches to the intersection index and mention the corresponding constructions of the biextension of Chow groups that will be given in the article.

The most explicit way to define the intersection index is to use the moving lemma and the definition of local multiplicities for proper intersections. Analogous to this is the explicit definition of the biextension P given in [7].

If $k = \mathbb{C}$, one can consider classes of algebraic cycles in Betti cohomology groups $H_B^{2p}(X(\mathbb{C}), \mathbb{Z})$ and then use the product between them and the push-forward map. Corresponding to this in [7] it was suggested that for $k = \mathbb{C}$, the biextension P should be equal to the pull-back via the Abel–Jacobi map of the Poincaré line bundle on the product of the corresponding dual intermediate Jacobians. This was partially proved in [22] by using the functorial properties of higher Chow groups and the regulator map to Deligne cohomology.

A different approach uses the Bloch–Quillen formula $CH^p(X) = H^p(X, \mathcal{K}_p)$, the product between cohomology groups of sheaves, the product between K -groups, and the push-forward for K -cohomology. Here \mathcal{K}_p is the Zariski sheaf associated to the presheaf given by the formula $U \mapsto K_p(U)$ for an open subset $U \subset X$. A corresponding approach to the biextension uses the pairing between complexes

$$R\Gamma(X, \mathcal{K}_p) \times R\Gamma(X, \mathcal{K}_q) \rightarrow k^*[-d]$$

for $p + q = d + 1$.

Finally, one can associate with each cycle $Z = \sum_i n_i Z_i$ an element $[\mathcal{O}_Z] = \sum_i n_i [\mathcal{O}_{Z_i}] \in K_0(X)$ and to use the natural pairing on $K_0(X)$, i.e., to define the intersection index by the formula $\text{rk} R\Gamma(X, \mathcal{O}_Z \otimes_{\mathcal{O}_X}^L \mathcal{O}_W)$. Analogous to this one considers

the determinant of cohomology $\det R\Gamma(X, \mathcal{O}_Z \otimes_{\mathcal{O}_X}^L \mathcal{O}_W)$ to get a biextension of Chow groups. This is a generalization of what was done for divisors on curves by P. Deligne in [11]. With this aim a new approach to J. Franke's Chow categories (see [12]) is developed. This approach uses a certain filtration “by codimension of support” on the Picard category of virtual coherent sheaves on a variety (see [11]).

One interprets the compatibility of the first definition of the intersection index with the second and the third one as the fact that the cycle maps

$$CH^p(X) \rightarrow H_B^{2p}(X(\mathbb{C}), \mathbb{Z}),$$

$$CH^p(X) \rightarrow H^{2p}(X, \mathcal{K}_p[-p])$$

commute with products and push-forwards. Note that the cycle maps are particular cases of canonical morphisms (regulators) from motivic cohomology to various cohomology theories. This approach explains quickly the comparison isomorphism between the corresponding biextensions. Besides this we give a more explicit and elementary proof of the comparison isomorphism in each case.

Question 1.1. What should be associated to a pair of cycles of codimensions p, q with $p + q = d + i$, $i \geq 2$? Presumably, when $i = 2$, one associates a $K_2(k)$ -gerbe.

The paper has the following structure. Sections 2.1-2.3 contain the description of various geometric constructions that are necessary for definitions of biextensions of Chow groups. In particular, in Section 2.4 we recall several facts from [16] about adelic resolution for sheaves of K -groups.

In Sections 3.1 and 3.2 general algebraic constructions of biextensions are discussed. In particular, we introduce the notion of a bisubgroup and give an explicit construction of a biextension induced by a pairing between complexes. Though these constructions are elementary and general, the author could not find any reference for them. As an example to the above notions, in Section 3.3 we consider the definition of the Poincaré biextension of dual complex compact tori by \mathbb{C}^* .

Sections 4.1-4.4 contain the constructions of biextensions of Chow groups according to different approaches to algebraic cycles. In Section 4.1 we recall from [7] an explicit construction of the biextension of Chow groups. This biextension is interpreted in terms of the pairing between higher Chow complexes (Proposition 4.2), as was suggested to the author by S. Bloch. In Section 4.2 for the complex base field, we consider the pull-back of the Poincaré biextension of dual intermediate Jacobians (Proposition 4.3). Section 4.3 is devoted to the construction of the biextension in terms of K -cohomology groups and a pairing between sheaves of K -groups (Proposition 4.9). We give an explicit description of this biextension in terms of the adelic resolution for sheaves of K -groups introduced in [16]. In Section 4.4 we construct a filtration on the Picard category of virtual coherent sheaves on a variety (Definition 4.10) and we define the biextension of Chow groups in terms of the determinant of cohomology (Proposition 4.29). In each section we establish a canonical isomorphism of the constructed biextension with the explicit biextension from [7] described firstly. In Sections 4.2 and 4.3 we give both explicit proofs and the proofs that use properties of the corresponding regulator maps. Finally, Section 4.5 gives

an explicit formula for the Weil pairing between torsion elements in $CH^*(X)_{\text{hom}}$. This can be considered as a generalization of the classical Weil's formula for divisors on a curve. Also, the equivalence of different constructions of biextensions of Chow groups implies the interpretation of the Weil pairing in terms of a certain Massey triple product.

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2 Preliminary results

2.1 Facts on higher Chow groups

All varieties below are defined over a fixed ground field k . For an equidimensional variety S , by $Z^p(S)$ denote the free abelian group generated by all codimension p irreducible subvarieties in S . For an element $Z = \sum n_i Z_i \in Z^p(S)$, by $|Z|$ denote the union of codimension p irreducible subvarieties Z_i in S such that $n_i \neq 0$. Let $S^{(p)}$ be the set of all codimension p schematic points on S .

Let us recall the definition of higher Chow groups (see [6]). We put $\Delta^n = \{\sum_{i=0}^n t_i = 1\} \subset \mathbb{A}^{n+1}$; note that Δ^\bullet is a cosimplicial variety. In particular, for each $m \leq n$, there are several face maps $\Delta^m \rightarrow \Delta^n$. For an equidimensional variety X , by $Z^p(X, n)$ denote the free abelian group generated by codimension p irreducible subvarieties in $X \times \Delta^n$ that meet the subvariety $X \times \Delta^m \subset X \times \Delta^n$ properly for any face $\Delta^m \subset \Delta^n$. The simplicial group $Z^p(X, \bullet)$ defines a homological type complex; by definition, $CH^p(X, n) = H_n(Z^p(X, \bullet))$ is the *higher Chow group* of X . Note that $CH^p(X, 0) = CH^p(X)$ and $CH^1(X, 1) = k[X]^*$ for a regular variety X (see op.cit.). For a projective morphism of varieties $f : X \rightarrow Y$, there is a push-forward morphism of complexes $f_* : Z^p(X, \bullet) \rightarrow Z^{p+\dim(Y)-\dim(X)}(Y, \bullet)$.

For an equidimensional subvariety $S \subset X$, consider the subcomplex $Z_S^p(X, \bullet) \subset Z^p(X, \bullet)$ generated by elements from $Z^p(X, n)$ whose support meets the subvariety $S \times \Delta^m \subset X \times \Delta^n$ properly for any face $\Delta^m \subset \Delta^n$. For a collection $\mathcal{S} = \{S_1 \dots, S_r\}$ of equidimensional subvarieties in X , we put $Z_{\mathcal{S}}^p(X, \bullet) = \cap_{i=1}^r Z_{S_i}^p(X, \bullet)$. The following moving lemma is proven in Proposition 2.3.1 from [8] and in [21]:

Lemma 2.1. *Provided that X is smooth over k and either projective or affine, the inclusion $Z_{\mathcal{S}}^p(X, \bullet) \subset Z^p(X, \bullet)$ is a quasiisomorphism for any \mathcal{S} as above.*

In particular, Lemma 2.1 allows to define the multiplication morphism

$$m \in \text{Hom}_{D^-(\text{Ab})}(Z^p(X, \bullet) \otimes_{\mathbb{Z}}^L Z^q(X, \bullet), Z^{p+q}(X, \bullet)).$$

Recall that a cycle $Z \in Z^p(X)$ is called *homologically trivial* if its class in the étale cohomology group $H_{\text{ét}}^{2p}(X_{\bar{k}}, \mathbb{Z}_l(p))$ is zero for any prime $l \neq \text{char}(k)$, where \bar{k} is the algebraic closure of the field k . Note that when $\text{char}(k) = 0$ the cycle Z is homologically trivial if and only if its class in the Betti cohomology group $H_B^{2p}(X_{\mathbb{C}}, \mathbb{Z})$ is zero after we

choose any model of X defined over \mathbb{C} . Denote by $CH^p(X)_{\text{hom}}$ the subgroup in $CH^p(X)$ generated by classes of homologically trivial cycles.

The following result is proved in [7], Lemma 1.

Lemma 2.2. *Let X be a smooth projective variety, $\pi : X \rightarrow \text{Spec}(k)$ be the structure morphism. Suppose that a cycle $Z \in Z^p(X)$ is homologically trivial; then the natural homomorphism $CH^{d+1-p}(X, 1) \xrightarrow{m(-\otimes Z)} CH^{d+1}(X, 1) \xrightarrow{\pi_*} CH^1(k, 1) = k^*$ is trivial.*

Remark 2.3. The proof of Lemma 2.2 uses the regulator map from higher Chow groups to Deligne cohomology if the characteristic is zero and to étale cohomology if the characteristic is positive.

Question 2.4. According to Grothendieck's standard conjectures, the statement of Lemma 2.2 (at least up to torsion in k^*) should be true if one replaces the homological triviality of the cycle W by the numerical one. Does there exist a purely algebraic proof of this fact that does not use Deligne cohomology or étale cohomology?

Remark 2.5. In Section 4.2 we give an analytic proof of Lemma 2.2 for complex varieties, which uses only general facts from the Hodge theory (see Lemma 4.8).

2.2 Facts on K_1 -chains

Let X be an equidimensional variety over the ground field k . We put $G^p(X, n) = \bigoplus_{\eta \in X^{(p)}} K_n(k(\eta))$ (in this section we use these groups only for $n = 0, 1, 2$). Elements of the group $G^{p-1}(X, 1)$ are called K_1 -chains. There are natural homomorphisms $\text{Tame} : G^{p-2}(X, 2) \rightarrow G^{p-1}(X, 1)$ and $\text{div} : G^{p-1}(X, 1) \rightarrow G^p(X, 0) = Z^p(X)$. Note that $\text{div} \circ \text{Tame} = 0$. The subgroup $\text{Im}(\text{Tame}) \subset G^{p-1}(X, 1)$ defines an equivalence on K_1 -chains; we call this a K_2 -equivalence on K_1 -chains. For a K_1 -chain $\{f_\eta\} \in G^{p-1}(X, 1)$, by $\text{Supp}(\{f_\eta\})$ denote the union of codimension $p-1$ irreducible subvarieties $\bar{\eta}$ in X such that $f_\eta \neq 1$.

Let S be an equidimensional subvariety; by $G_S^{p-1}(X, 1)$ denote the group of K_1 -chains $\{f_\eta\}$ such that for any $\eta \in X^{(p-1)}$, either $f_\eta = 1$, or the closure $\bar{\eta}$ and the support $|\text{div}(f_\eta)|$ meet S properly. For a collection $\mathcal{S} = \{S_1, \dots, S_r\}$ of equidimensional subvarieties in X , we put $G_{\mathcal{S}}^{p-1}(X, 1) = \bigcap_{i=1}^r G_{S_i}^{p-1}(X, 1)$.

Define the homomorphism $N : Z^p(X, 1) \rightarrow G^{p-1}(X, 1)$ as follows. Let Y be an irreducible subvariety in $X \times \Delta^1$ that meets properly both faces $X \times \{(0, 1)\}$ and $X \times \{(1, 0)\}$ (recall that $\Delta^1 = \{t_0 + t_1 = 1\} \subset \mathbb{A}^2$). By p_X and p_{Δ^1} denote the projections from $X \times \Delta^1$ to X and Δ^1 , respectively. If the morphism $p_X : Y \rightarrow X$ is not generically finite onto its image, then we put $N(Y) = 0$. Otherwise, let $\eta \in X^{(p-1)}$ be the generic point of $p_X(Y)$; we put $f_\eta = (p_X)_*(p_{\Delta^1}^*(t_1/t_0)) \in k(\eta)^*$ and $N(Y) = f_\eta \in G^{p-1}(X, 1)$. We extend the homomorphism N to $Z^p(X, 1)$ by linearity.

Conversely, given a point $\eta \in X^{(p-1)}$ and a rational function $f_\eta \in k(\eta)^*$ such that $f_\eta \neq 1$, let $\Gamma(f_\eta)$ be the closure of the graph of the rational map $(\frac{1}{1+f_\eta}, \frac{f_\eta}{1+f_\eta}) : \bar{\eta} \dashrightarrow \Delta^1 \subset \mathbb{A}^2$. This defines the map of sets $\Gamma : G^{p-1}(X, 1) \rightarrow Z^p(X, 1)$ such that $N \circ \Gamma$ is the

identity. For an element $Y \in Z^p(X, 1)$, we have $\text{div}(N(Y)) = d(Y) \in Z^p(X)$, where d denotes the differential in the complex $Z^p(X, \bullet)$.

Given an equidimensional subvariety $S \subset X$, it is easy to check that $\Gamma(G_S^{p-1}(X, 1)) \subset Z_S^p(X, 1)$ and $N(Z_S^p(X, 1)) \subset G_S^{p-1}(X, 1)$.

Lemma 2.6. *Suppose that X is smooth over k and either projective or affine. Let $\mathcal{S} = \{S_1, \dots, S_r\}$ be a collection of equidimensional closed subvarieties in X , $\{f_\eta\} \in G^{p-1}(X, 1)$ be a K_1 -chain such that the support $|\text{div}(\{f_\eta\})|$ meets S_i properly for all i , $1 \leq i \leq r$; then there exists a K_1 -chain $\{g_\eta\} \in G_S^{p-1}(X, 1)$ such that $\{g_\eta\}$ is K_2 -equivalent to $\{f_\eta\}$.*

Proof. Denote by d the differential in the complex $Z^p(X, \bullet)$. Since $\text{div}(\{f_\eta\}) \in Z_S^p(X, 0)$, by Lemma 2.1, there exists an element $Y' \in Z_S^p(X, 1)$ such that $d(Y') = \text{div}(\{f_\eta\}) = d(\Gamma(\{f_\eta\}))$. Again by Lemma 2.1, there exists an element $Y'' \in Z_S^p(X, 1)$ such that $d(Y'') = 0$ and $Y'' + Y' - \Gamma(\{f_\eta\}) = d(\tilde{Y})$ for some $\tilde{Y} \in Z^p(X, 2)$.

Recall that $(N \circ d)(Z^p(X, 2)) \subset \text{Im}(\text{Tame}) \subset G^{p-1}(X, 1)$, see [23], Remark on p. 13 for more details. Therefore, the K_1 -chain $\{g_\eta\} = N(Y'' + Y') \in G_S^{p-1}(X, 1)$ is K_2 -equivalent to $\{f_\eta\}$. \square

Corollary 2.7. *In notations from Lemma 2.6 let $Z = \text{div}(\{f_\eta\})$ and suppose that $\text{codim}_Z(Z \cap S_i) \geq n_i$ for all i , $1 \leq i \leq r$ (in particular, $n_i \leq \text{codim}_X(S_i)$); then there exists a K_1 -chain $\{g_\eta\} \in G^{p-1}(X, 1)$ such that $\{g_\eta\}$ is K_2 -equivalent to $\{f_\eta\}$, $\text{codim}_Y(Y \cap S_i) \geq n_i$ for all i , $1 \leq i \leq r$, where $Y = \text{Supp}(\{g_\eta\})$, and for each $\eta \in Z^{p-1}(X)$, we have $\text{codim}_{\text{div}(g_\eta)}(\text{div}(g_\eta) \cap S_i) \geq n_i$ for all i , $1 \leq i \leq r$.*

Proof. We claim that for each i , $1 \leq i \leq r$, there exists an equidimensional subvariety $S'_i \subset X$ of codimension n_i such that $S'_i \supset S_i$ and Z meets S'_i properly. By Lemma 2.6, this immediately implies the needed statement.

For each i , $1 \leq i \leq r$, we prove the existence of S'_i by induction on n_i . Suppose that $n_i = 1$. Then there exists an effective reduced divisor $H \subset X$ such that $H \supset S_i$ and H meets Z properly: to construct such divisor we have to choose a closed point on each irreducible component of Z outside of S_i and take an arbitrary H that does not contain any of these points and such that $H \supset S_i$.

Now let us do the induction step from $n_i - 1$ to n_i . Let $\tilde{S}_i \subset X$ be an equidimensional subvariety that satisfies the needed condition for $n_i - 1$. For each irreducible component of \tilde{S}_i choose a closed point on it outside of S_i . Also, for each irreducible component of $\tilde{S}_i \cap Z$ choose a closed point on it outside of S_i . Thus we get a finite set T of closed points in X outside of S_i . Let H be an effective reduced divisor on X such that $H \supset S_i$ and $H \cap T = \emptyset$; then we put $S'_i = \tilde{S}_i \cap H$. \square

Let W be a codimension q cycle on X , Y be an irreducible subvariety of codimension $d - q$ in X that meets $|W|$ properly, and let f be a rational function on Y such that $\text{div}(f)$ does not intersect with $|W|$. We put $f(Y \cap W) = \prod_{x \in Y \cap |W|} \text{Nm}_{k(x)/k}(f^{(Y, W; x)}(x))$, where $(Y, W; x)$ is the intersection index of Y with the cycle W at a point $x \in Y \cap |W|$.

Lemma 2.8. *Let X be a smooth projective variety over k and let $p+q = d+1$; consider cycles $Z \in Z^p(X)$, $W \in Z^q(X)$, and K_1 -chains $\{f_\eta\} \in G_{|W|}^{p-1}(X, 1)$, $\{g_\xi\} \in G_{|Z|}^{q-1}(X, 1)$ such that $\text{div}(\{f_\eta\}) = Z$, $\text{div}(\{g_\xi\}) = W$. Then we have $\prod_\eta f_\eta(\bar{\eta} \cap W) = \prod_\xi g_\xi(Z \cap \bar{\xi})$.*

Proof. Note that the left hand side depends only in the K_2 -equivalence class of the K_1 -chain $\{f_\eta\}$. Therefore by Lemma 2.6, we may assume that $\{f_\eta\} \in G_{\mathcal{S}}^{p-1}(X, 1)$, where $\mathcal{S} = \{\text{Supp}(\{g_\xi\}), \cup_\xi |\text{div}(g_\xi)|\}$. For each pair $\eta \in \text{Supp}(\{f_\eta\})$, $\xi \in \text{Supp}(\{g_\xi\})$, let $C_{\eta\xi}^\alpha$ be an irreducible component of the intersection $\bar{\eta} \cap \bar{\xi}$ and let $n_{\eta\xi}^\alpha$ be the intersection index of the subvarieties $\bar{\eta}$ and $\bar{\xi}$ at the irreducible curve $C_{\eta\xi}^\alpha$. By condition, for all η, ξ, α as above, the restrictions $f_{\eta\xi}^\alpha = f_\eta|_{C_{\eta\xi}^\alpha}$ and $g_{\eta\xi}^\alpha = g_\xi|_{C_{\eta\xi}^\alpha}$ are well defined as rational functions on the irreducible curve $C_{\eta\xi}^\alpha$. It follows that $\prod_\eta f_\eta(\bar{\eta} \cap W) = \prod_{\eta, \xi, \alpha} f_{\eta\xi}^\alpha(\text{div}(g_{\eta\xi}^\alpha))^{n_{\eta\xi}^\alpha}$; thus we conclude by the classical Weil reciprocity law for curves. \square

Remark 2.9. The same reasoning as in the proof of Lemma 2.8 is explained in a slightly different language in the proof of Proposition 3 from [7].

Lemma 2.10. *Let X be a smooth projective variety over k . Suppose that a cycle $W \in Z^q(X)$ is homologically trivial; then for any K_1 -chain $\{f_\eta\} \in G_{|W|}^{d-q}(X, 1)$ with $\text{div}(\{f_\eta\}) = 0$, we have $\prod_\eta f_\eta(\bar{\eta} \cap W) = 1$.*

Proof. By condition, $\Gamma(\{f_\eta\}) \in Z_{|W|}^{d-q}(X, 1)$. Keeping in mind the explicit formula for product in higher Chow groups for cycles in general position (see [6]), we see that $((\pi_*) \circ m)(\Gamma(\{f_\eta\}) \otimes W) \in k^*$ is well defined and coincides with $\prod_\eta f_\eta(\bar{\eta} \cap W)$. Hence we conclude by Lemma 2.2. \square

Remark 2.11. If $q = d$, then Lemma 2.10 is trivial. An elementary proof of Lemma 2.10 for the case $q = 1$ can be found in [16].

2.3 Facts on the Abel–Jacobi map

Notions and results of this section are used in Section 4.2.

Let X be a complex smooth variety of dimension d . By A_X^n denote the group of complex valued smooth differential forms on X of degree n . Let $F^p A_X^n$ be the subgroup in A_X^n that consists of all differential forms with at least p “ dz_i ”. If X is projective, then the classical Hodge theory implies $H^n(F^p A_X^\bullet) = F^p H^n(X, \mathbb{C})$, where we consider the Hodge filtration in the right hand side.

Let S be a closed subvariety in a smooth projective variety X ; then the notation $\eta \in F_{\log}^p A_{X \setminus S}^n$ means that there exists a smooth projective variety X' together with a birational morphism $f : X' \rightarrow X$ such that $D = f^{-1}(S)$ is a normal crossing divisor on X' , f induces an isomorphism $X' \setminus f^{-1}(S) \rightarrow X \setminus S$, and $f^* \eta \in F^p A_{X'}^n \langle D \rangle$, where $A_{X'}^n \langle D \rangle$ is the group of complex valued smooth differential forms on $X' \setminus D$ of degree n with logarithmic singularities along D . Recall that any class in $F^p H^n(X \setminus S, \mathbb{C})$ can be represented by a closed form $\eta \in F_{\log}^p A_{X \setminus S}^n$ (see [10]).

In what follows the variety X is supposed to be projective. By $H^*(X, \mathbb{Z})$ we often mean the image of this group in $H^*(X, \mathbb{C})$. Recall that the p -th intermediate Jacobian $J^{2p-1}(X)$ of X is the compact complex torus given by the formula

$$\begin{aligned} J^{2p-1}(X) &= H^{2p-1}(X, \mathbb{C}) / (H^{2p-1}(X, \mathbb{Z}) + F^p H^{2p-1}(X, \mathbb{C})) = \\ &= F^{d-p+1} H^{2d-2p+1}(X, \mathbb{C})^* / H_{2d-2p+1}(X, \mathbb{Z}). \end{aligned}$$

Let Z be a homologically trivial algebraic cycle on X of codimension p . Then there exists a differentiable singular chain Γ of dimension $2d - 2p + 1$ with $\partial\Gamma = Z$, where ∂ denotes the differential in the complex of singular chains on X . Consider a closed differential form $\omega \in F^{d-p+1} A_X^{2d-2p+1}$. It can be easily checked that the integral $\int_\Gamma \omega$ depends only on the cohomology class $[\omega] \in F^{d-p+1} H^{2d-2p+1}(X, \mathbb{C}) = H^{2d-2p+1}(F^{d-p+1} A_X^\bullet)$ of ω . Thus the assignment

$$Z \mapsto \{[\omega] \mapsto \int_\Gamma \omega\}$$

defines a homomorphism $AJ : Z^p(X)_{\text{hom}} \rightarrow J^{2p-1}(X)$, which is called the *Abel–Jacobi map*.

We give a slightly different description of the Abel–Jacobi map. Recall that there is an exact sequence of integral mixed Hodge structures:

$$0 \rightarrow H^{2p-1}(X)(p) \rightarrow H^{2p-1}(X \setminus |Z|)(p) \xrightarrow{\partial_Z} H_{2d-2p}(|Z|) \rightarrow H^{2p}(X)(p).$$

Recall that $H_{2d-2p}(|Z|) = \oplus_i \mathbb{Z}(0)$, where the sum is taken over all irreducible components in $|Z| = \cup_i Z_i$. Thus the cycle Z defines an element $[Z] \in F^0 H_{2d-2p}(|Z|, \mathbb{C})$ with a trivial image in the group $H^{2p}(X, \mathbb{Z}(p))$. Hence there exists a closed differential form $\eta \in F_{\log}^p A_{X \setminus |Z|}^{2p-1}$ such that $\partial_Z([(2\pi i)^p \eta]) = [Z]$. The difference $PD[\Gamma] - [\eta]$ defines a unique element in the group $H^{2p-1}(X, \mathbb{C})$, where $PD : H_*(X, |Z|; \mathbb{Z}) \rightarrow H^{2d-*}(X \setminus |Z|, \mathbb{Z})$ is the canonical isomorphism induced by Poincaré duality.

Lemma 2.12. *The image of $PD[\Gamma] - [\eta] \in H^{2p-1}(X, \mathbb{C})$ in the intermediate Jacobian $J^{2p-1}(X)$ is equal to the image of Z under the Abel–Jacobi map.*

Proof. Consider a closed differential form $\omega \in F^{d-p+1} A_X^{2d-2p+1}$. Since $\dim(Z) = d - p$, the form ω also defines the class $[\omega] \in F^{d-p+1} H^{2d-2p+1}(X, |Z|; \mathbb{C})$. Denote by

$$(\cdot, \cdot) : H^*(X \setminus |Z|, \mathbb{C}) \times H^{2d-*}(X, |Z|; \mathbb{C}) \rightarrow \mathbb{C}$$

the natural pairing. Then we have $(PD[\Gamma] - [\eta], [\omega]) = (PD[\Gamma], [\omega]) = \int_\Gamma \omega$; this proves the needed result. \square

Remark 2.13. It follows from Lemma 2.12 that $AJ(Z) = 0$ if and only if there exists an element $\alpha \in F^p H^{2p-1}(X \setminus |Z|, \mathbb{C}) \cap H^{2p-1}(X \setminus |Z|, \mathbb{Z}(p))$ such that $\partial_Z(\alpha) = [Z]$.

Example 2.14. Suppose that $X = \mathbb{P}^1$, $Z = \{0\} - \{\infty\}$. Let z be a coordinate on \mathbb{P}^1 and Γ be a smooth generic path on \mathbb{P}^1 such that $\partial\Gamma = \{0\} - \{\infty\}$; then we have $\alpha = [\frac{dz}{z}] = 2\pi i PD[\Gamma] \in F^1 H^1(X \setminus |Z|, \mathbb{C}) \cap H^1(X \setminus |Z|, \mathbb{Z}(1))$ and $\partial_Z(\alpha) = Z$.

Lemma 2.15. *Suppose that the cycle Z is rationally trivial; then $AJ(Z) = 0$.*

Proof 1. By linearity, it is enough to consider the case when $Z = \text{div}(f)$, $f \in \mathbb{C}(Y)^*$, $Y \subset X$ is an irreducible subvariety of codimension $p - 1$. Let \tilde{Y} be the closure of the graph of the rational function $f : Y \dashrightarrow \mathbb{P}^1$ and let $p : \tilde{Y} \rightarrow X$ be the natural map. In [17] it was shown that the following map is holomorphic:

$$\varphi : \mathbb{P}^1 \rightarrow J^{2p-1}(X), z \mapsto AJ(p_* f^{-1}(\{z\} - \{\infty\}));$$

therefore φ is constant and $AJ(Z) = \varphi(0) = \{0\}$.

Proof 2. Let α be as in Example 2.14; then by Lemma 2.17 with $X_1 = \mathbb{P}^1$, $X_2 = X$, $C = \tilde{Y}$, $Z_1 = \{0, \infty\}$, $W_2 = \emptyset$, we have $[\tilde{Y}]^* \alpha \in F^p H^{2p-1}(X \setminus |Z|, \mathbb{C}) \cap H^{2p-1}(X \setminus |Z|, \mathbb{Z}(p))$ and $\partial_Z([\tilde{Y}]^* \alpha) = Z$; thus we conclude by Remark 2.13. \square

Remark 2.16. If there is a differentiable triangulation of the closed subset $p(f^{-1}(\gamma)) \subset X$, then we have a well defined class $[p(f^{-1}(\gamma))] \in H_{2d-2p+1}(X, |Z|; \mathbb{Z})$ and $\alpha = (2\pi i)^p PD[p(f^{-1}(\gamma))]$.

In particular, we see that the Abel–Jacobi map factors through Chow groups. In what follows we consider the induced map $AJ : CH^p(X)_{\text{hom}} \rightarrow J^{2p-1}(X)$.

In the second proof of Lemma 2.15 we have used the following simple fact. Let X_1 and X_2 be two complex smooth projective varieties of dimensions d_1 and d_2 , respectively. Suppose that $C \subset X_1 \times X_2$, $Z_1 \subset X_1$, and $W_2 \subset X_2$ are closed subvarieties; we put $Z_2 = \pi_2(\pi_1^{-1}(Z_1) \cap C)$, $W_1 = \pi_1(\pi_2^{-1}(W_2) \cap C)$, where $\pi_i : X_1 \times X_2 \rightarrow X_i$, $i = 1, 2$ denote the natural projections.

Lemma 2.17. (i) *Let c be the codimension of C in $X_1 \times X_2$; then there is a natural morphism of integral mixed Hodge structures*

$$[C]^* : H^*(X_1 \setminus Z_1, W_1) \rightarrow H^{*+2c-2d_1}(X_2 \setminus Z_2, W_2)(c - d_1).$$

(ii) *For $i = 1, 2$, let p_i be the codimension of Z_i in X_i . Suppose that C meets $\pi_1^{-1}(Z_1)$ properly, the intersection $\pi_1^{-1}(Z_1) \cap \pi_2^{-1}(W_2) \cap C$ is empty, and $p_1 + c - d_1 = p_2$; then $Z_1 \cap W_1 = Z_2 \cap W_2 = \emptyset$ and the following diagram commutes:*

$$\begin{array}{ccc} H^{2p_1-1}(X_1 \setminus Z_1, W_1)(p_1) & \xrightarrow{\partial_{Z_1}} & H_{2d_1-2p_1}(Z_1) = Z^0(Z_1) \otimes \mathbb{Z}(0) \\ \downarrow [C]^* & & \downarrow \pi_2(C \cap \pi_1^{-1}(\cdot)) \\ H^{2p_2-1}(X_2 \setminus Z_2, W_2)(p_2) & \xrightarrow{\partial_{Z_2}} & H_{2d_2-2p_2}(Z_2) = Z^0(Z_2) \otimes \mathbb{Z}(0), \end{array}$$

where the right vertical arrow is defined via the corresponding natural homomorphisms of groups of algebraic cycles.

Proof. The needed morphism $[C]^*$ is the composition of the following natural morphisms

$$\begin{aligned} H^*(X_1 \setminus Z_1, W_1) &\rightarrow H^*((X_1 \times X_2) \setminus \pi_1^{-1}(Z_1), \pi_2^{-1}(W_2) \cap C) \xrightarrow{\cap [C]} \\ &\xrightarrow{\cap [C]} H_{C \setminus \pi_1^{-1}(Z_1)}^{*+2c}((X_1 \times X_2) \setminus \pi_1^{-1}(Z_1), \pi_2^{-1}(W_2) \cap C)(c) = \end{aligned}$$

$$\begin{aligned}
&= H_{2d_1+2d_2-* -2c}(C \setminus \pi_2^{-1}(W_2), \pi_1^{-1}(Z_1) \cap C) = \\
&= H_{C \setminus \pi_1^{-1}(Z_1)}^{*+2c}((X_1 \times X_2) \setminus (\pi_1^{-1}(Z_1) \cap C), \pi_2^{-1}(W_2))(c) \rightarrow \\
&\rightarrow H^{*+2c}((X_1 \times X_2) \setminus (\pi_1^{-1}(Z_1) \cap Y), \pi_2^{-1}(W_2))(c) \rightarrow H^{*+2c-2d_1}(X_2 \setminus Z_2, W_2)(c - d_1),
\end{aligned}$$

where the first morphism is the natural pull-back map, the second one is multiplication by the fundamental class $[C] \in H_C^{2c}(X_1 \times X_2)(c)$, the equalities in the middle follow from the excision property, and the last morphism is the push-forward map. The second assertion follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
H^*((X_1 \times X_2) \setminus (\pi_1^{-1}(Z_1) \cap C), \pi_2^{-1}(W_2)) & \rightarrow & H_{2d_1+2d_2-1-*}(\pi_1^{-1}(Z_1) \cap C)(-c - p_1) \\
\downarrow & & \downarrow \\
H^{*-2d_1}(X_2 \setminus Z_2, W_2)(-d_1) & \rightarrow & H_{2d_1+2d_2-1-*}(Z_2)(-c - p_1).
\end{array}$$

□

2.4 Facts on K -adeles

Notions and results of this section are used in Section 4.3.

Let X be an equidimensional variety over the ground field k . Let \mathcal{K}_n be the sheaf on X associated to the presheaf given by the formula $U \mapsto K_n(U)$, where $K_n(-)$ is the Quillen K -group and U is an open subset in X . Zariski cohomology groups of the sheaves \mathcal{K}_n are called *K -cohomology groups*. When it will be necessary for us to point out the underline variety, we will use notation \mathcal{K}_n^X for the defined above sheaf \mathcal{K}_n on X .

Recall that in notations from Section 2.2, for all integers $n \geq 1, p \geq 0$, there are natural homomorphisms $d : G^p(X, n) \rightarrow G^{p+1}(X, n-1)$ such that $d^2 = 0$. Thus for each $n \geq 0$, there is a complex $Gers(X, n)^\bullet$, where $Gers(X, n)^p = G^p(X, n-p)$; this complex is called the *Gersten complex*. Note that the homomorphisms $d : G^{p-1}(X, 1) \rightarrow G^p(X, 0)$ and $d : G^{p-2}(X, 2) \rightarrow G^{p-1}(X, 1)$ coincide with the homomorphisms div and Tame , respectively. For a projective morphism of varieties $f : X \rightarrow Y$, there is a push-forward morphism of complexes $f_* : Gers(X, n)^\bullet \rightarrow Gers(Y, n + \dim(Y) - \dim(X))^\bullet[\dim(Y) - \dim(X)]$.

In what follows we suppose that X is *smooth* over the field k . By results of Quillen (see [27]), for each $n \geq 0$, there is a canonical isomorphism between the classes of the complexes $Gers(X, n)^\bullet$ and $R\Gamma(X, \mathcal{K}_n)$ in the derived category $D^b(\mathcal{A}b)$. In particular, there is a canonical isomorphism $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ for all $p \geq 0$. The last statement is often called the Bloch–Quillen formula.

There is a canonical product between the sheaves of K -groups, induced by the product in K -groups themselves. However, the Gersten complex *is not multiplicative*, i.e., there is no a product between Gersten complexes that would correspond to the product between sheaves \mathcal{K}_n : otherwise there would exist an intersection theory for algebraic cycles without taking them modulo rational equivalence. Explicitly, there is no a morphism of complexes

$$Gers(X, m)^\bullet \otimes_{\mathbb{Z}} Gers(X, n)^\bullet \rightarrow Gers(X, m+n)^\bullet$$

that would correspond to the natural product between cohomology groups

$$H^\bullet(X, \mathcal{K}_m) \otimes H^\bullet(X, \mathcal{K}_n) \rightarrow H^\bullet(X, \mathcal{K}_{m+n}).$$

Therefore if one would like to work explicitly with the pairing of objects in the derived category

$$R\Gamma(X, \mathcal{K}_m) \otimes_{\mathbb{Z}}^L R\Gamma(X, \mathcal{K}_n) \rightarrow R\Gamma(X, \mathcal{K}_{m+n}),$$

then a natural way would be to use a different resolution rather than the Gersten complex. There are general multiplicative resolutions of sheaves, for example a Godement resolution, but it does not see the Bloch–Quillen isomorphism. In particular, there is no explicit quasiisomorphism between the Godement and the Gersten complexes.

In [16] the author proposed another way to construct resolutions for a certain class of abelian sheaves on smooth algebraic varieties, namely, the *adelic resolution*. This class of sheaves includes the sheaves \mathcal{K}_n . It is *multiplicative* and there is an *explicit quasiisomorphism* from the adelic resolution to the Gersten complex.

Remark 2.18. Analogous adelic resolutions for coherent sheaves on algebraic varieties have been first introduced by A.N. Parshin (see [26]) in the two-dimensional case, and then developed by A.A. Beilinson (see [1]) and A. Huber (see [19]) in the higher-dimensional case.

Remark 2.19. When the paper was finished, the author discovered that a similar but more general construction of a resolution for sheaves on algebraic varieties was independently done in [4, Section 4.2.2] by A. A. Beilinson and V. Vologodsky.

Let us briefly recall several notions and facts from [16]. A *non-degenerate flag of length p* on X is a sequence of schematic points $\eta_0 \dots \eta_p$ such that $\eta_{i+1} \in \overline{\eta_i}$ and $\eta_{i+1} \neq \eta_i$ for all i , $0 \leq i \leq p-1$. For $n, p \geq 0$, there are *adelic groups*

$$\mathbf{A}(X, \mathcal{K}_n)^p \subset \prod_{\eta_0 \dots \eta_p} K_n(\mathcal{O}_{X, \eta_0}),$$

where the product is taken over all non-degenerate flags of length p , \mathcal{O}_{X, η_0} is the local ring of the scheme X at a point η_0 , and the subgroup $\mathbf{A}(X, \mathcal{K}_n)^p$ is defined by certain explicit conditions concerning “singularities” of elements in K -groups. Elements of the adelic groups are called *K-adeles* or just *adeles*. Explicitly, an adele $f \in \mathbf{A}(X, \mathcal{K}_n)^p$ is a collection $f = \{f_{\eta_0 \dots \eta_p}\}$ of elements $f_{\eta_0 \dots \eta_p} \in K_n(\mathcal{O}_{X, \eta_0})$ that satisfies certain conditions.

Example 2.20. If X is a smooth curve over the ground field k , then

$$\mathbf{A}(X, \mathcal{K}_n)^0 = K_n(k(X)) \times \prod_{x \in X} K_n(\mathcal{O}_{X, x}),$$

where the product is taken over all closed points $x \in X$, and an adele $f \in \mathbf{A}(X, \mathcal{K}_n)^1$ is a collection $f = \{f_{Xx}\}$, $f_{Xx} \in K_n(k(X))$, such that $f_{Xx} \in K_n(\mathcal{O}_{X, x})$ for almost all $x \in X$ (this is the restricted product condition in this case). The apparent similarity with classical adele and idele groups explains the name of the notion.

There is a differential

$$d : \mathbf{A}(X, \mathcal{K}_n)^p \rightarrow \mathbf{A}(X, \mathcal{K}_n)^{p+1}$$

defined by the formula

$$(df)_{\eta_0 \dots \eta_p} = \sum_{i=0}^p (-1)^i f_{\eta_0 \dots \hat{\eta}_i \dots \eta_p},$$

where the hat over η_i means that we omit a point η_i . It can be easily seen that $d^2 = 0$, so one gets an *adelic complex* $\mathbf{A}(X, \mathcal{K}_n)^\bullet$. There is a canonical morphism of complexes $\nu_X : \mathbf{A}(X, \mathcal{K}_n)^\bullet \rightarrow \text{Gers}(X, n)^\bullet$. There is also an adelic complex of flabby sheaves $\underline{\mathbf{A}}(X, \mathcal{K}_n)^\bullet$ given by the formula $\underline{\mathbf{A}}(X, \mathcal{K}_n)^p(U) = \mathbf{A}(U, \mathcal{K}_n)^p$ and a natural morphism of complexes of sheaves $\mathcal{K}_n[0] \rightarrow \underline{\mathbf{A}}(X, \mathcal{K}_n)^\bullet$.

In what follows we suppose that the ground field k is *infinite and perfect*.

Lemma 2.21. ([16, Theorem 3.34]) *The complex of sheaves $\underline{\mathbf{A}}(X, \mathcal{K}_n)^\bullet$ is a flabby resolution for the sheaf \mathcal{K}_n . The morphism ν_X is a quasiisomorphism; in particular, this induces a canonical isomorphism between the classes of the complexes $\mathbf{A}(X, \mathcal{K}_n)^\bullet$ and $R\Gamma(X, \mathcal{K}_n)$ in the derived category $D^b(\mathcal{A}b)$.*

In particular, there is a canonical isomorphism

$$H^p(X, \mathcal{K}_n) = H^p(\mathbf{A}(X, \mathcal{K}_n)^\bullet).$$

The main advantages of adelic complexes are the contravariancy and the multiplicativity properties.

Lemma 2.22. ([16, Remark 2.12])

(i) *Given a morphism $f : X \rightarrow Y$ of smooth varieties over k , for each $n \geq 0$, there is a morphism of complexes*

$$f^* : \mathbf{A}(Y, \mathcal{K}_n^Y)^\bullet \rightarrow \mathbf{A}(X, \mathcal{K}_n^X)^\bullet;$$

this morphism agrees with the natural morphism $\text{Hom}_{D^b(\mathcal{A}b)}(R\Gamma(Y, f_ \mathcal{K}_n^X), R\Gamma(X, \mathcal{K}_n^X))$ and the morphism $\mathcal{K}_n^Y \rightarrow f_* \mathcal{K}_n^X$ of sheaves on Y .*

(ii) *For all $p, q \geq 0$, there is a morphism of complexes*

$$m : \mathbf{A}(X, \mathcal{K}_p)^\bullet \otimes \mathbf{A}(X, \mathcal{K}_q)^\bullet \rightarrow \mathbf{A}(X, \mathcal{K}_{p+q})^\bullet;$$

this morphism agrees with the multiplication morphism

$$m \in \text{Hom}_{D^b(\mathcal{A}b)}(R\Gamma(X, \mathcal{K}_p) \otimes_{\mathbb{Z}}^L R\Gamma(X, \mathcal{K}_q), R\Gamma(X, \mathcal{K}_{p+q})).$$

In what follows we recollect some technical notions and facts that are used in calculations with elements of the adelic complex. The idea is to associate with each cocycle in the Gersten complex, an explicit cocycle in the adelic resolution with the same class in K -cohomology. The adelic cocycle should be good enough so that it would be easy to calculate its product with other (good) adelic cocycles. This allows to analyze explicitly the interrelation between the product on complexes $R\Gamma(X, \mathcal{K}_n)$ and the Bloch–Quillen isomorphism.

For any equidimensional subvariety $Z \subset X$ of codimension p in X , there is a notion of a *patching system* $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ for Z on X , where Z_r^1 and Z_r^2 are equidimensional subvarieties in X of codimension r such that the system $\{Z_r^{1,2}\}$ satisfies certain properties; in particular, we have:

- (i) the varieties Z_r^1 and Z_r^2 have no common irreducible components for all r , $1 \leq r \leq p-1$;
- (ii) the variety Z is contained in both varieties Z_{p-1}^1 and Z_{p-1}^2 , and the variety $Z_r^1 \cup Z_r^2$ is contained in both varieties Z_{r-1}^1 and Z_{r-1}^2 for all r , $2 \leq r \leq p-1$.

Remark 2.23. What we call here a patching system is what is called in [16] *a patching system with the freedom degree at least zero*.

Lemma 2.24. ([16, Remark 3.32])

- (i) Suppose that $Z \subset X$ is an equidimensional subvariety of codimension p in X ; then there exists a patching system $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ for Z on X such that each irreducible component of Z_{p-1}^1 and Z_{p-1}^2 contains some irreducible component of Z and for any r , $1 \leq r \leq p-2$, each irreducible component of Z_r^1 and Z_r^2 contains some irreducible component of $Z_{r+1}^1 \cup Z_{r+1}^2$;
- (ii) given an equidimensional subvariety $W \subset X$ that meets Z properly, one can require in addition that no irreducible component of $W \cap Z_r^1$ is contained in Z_r^2 for all r , $1 \leq r \leq p-1$.

Remark 2.25. If an equidimensional subvariety $W \subset X$ meets Z properly and the patching system $\{Z_r^{1,2}\}$ satisfies the condition (i) from Lemma 2.24, then W meets Z_r^i properly for $i = 1, 2$ and all r , $1 \leq r \leq p-1$.

Suppose that $\{f_\eta\} \in \text{Gers}(X, n)^p$ is a cocycle in the Gersten complex and let Z be the support of $\{f_\eta\}$. Given a patching system $Z_r^{1,2}$, $1 \leq r \leq p-1$ for Z on X , there is a notion of a *good cocycle* $[\{f_\eta\}] \in \mathbf{A}(X, \mathcal{K}_n)^p$ with respect to the patching system $\{Z_r^{1,2}\}$. In particular, we have $d[\{f_\eta\}] = 0$, $\nu_X[\{f_\eta\}] = \{f_\eta\}$, and $i_U^*[\{f_\eta\}] = 0 \in \mathbf{A}(U, \mathcal{K}_n)^p$, where $i_U : U = X \setminus Z \hookrightarrow X$ is the open embedding. Thus the good cocycle $[\{f_\eta\}]$ is a cocycle in the adelic complex $\mathbf{A}(X, \mathcal{K}_n)^\bullet$, which represents the cohomology class in $H^p(X, \mathcal{K}_n)$ of the Gersten cocycle $\{f_\eta\}$. In addition, $[\{f_\eta\}]$ satisfies certain properties that allow to consider explicitly its products with other cocycles.

Lemma 2.26. ([16, Claim 3.47]) Let $\{f_\eta\}$, Z , and $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ be as above; then there exists a good cocycle for $\{f_\eta\}$ with respect to the patching system $Z_r^{1,2}$.

Consider a cycle $Z \in Z^p(X)$; suppose that a K_1 -chain $\{f_\eta\} \in G^{p-1}(X, 1)$ is such that $\text{div}(\{f_\eta\}) = Z$ and a K -adele $[Z] \in \mathbf{A}(X, \mathcal{K}_p)^p$ is such that $d[Z] = 0$, $\nu_X([Z]) = Z$, and $i_U^*[Z] = 0 \in \mathbf{A}(U, \mathcal{K}_p)^p$, where $i_U : U = X \setminus |Z| \hookrightarrow X$ is the open embedding.

Lemma 2.27. ([16, Lemma 3.48]) *In the above notations, let $Y = \text{Supp}(\{f_\eta\})$ and let $\{Y_r^{1,2}\}$ be a patching system for Y on X ; then there exists a K -adele $[\{f_\eta\}] \in \mathbf{A}(X, \mathcal{K}_p)^{p-1}$ such that $d[\{f_\eta\}] = [Z]$, $\nu_X[\{f_\eta\}] = \{f_\eta\}$, and $i_U^*[\{f_\eta\}] \in \mathbf{A}(U, \mathcal{K}_p)^{p-1}$ is a good cocycle with respect to the restriction of the patching system $\{Y_r^{1,2}\}$ to U .*

For a cycle $Z \in Z^p(X)$, let $\{Z_r^{1,2}\}$, $1 \leq r \leq p-1$ be a patching system for $|Z|$ on X and $[Z] \in \mathbf{A}(X, \mathcal{K}_p)^p$ be a good cocycle for Z with respect to the patching system $\{Z_r^{1,2}\}$. Given a K_1 -chain $\{f_z\} \in G^p(X, 1)$ with support on $|Z|$ and $\text{div}(\{f_z\}) = 0$, let $[\{f_z\}] \in \mathbf{A}(X, p+1)^p$ be a good cocycle for $\{f_z\}$ with respect to the patching system $\{Z_r^{1,2}\}$. Let $\{W_s^{1,2}\}$, $1 \leq s \leq q-1$, $[W]$, and $[\{g_w\}] \in \mathbf{A}(X, q+1)^q$ be the analogous objects for a cycle $W \in Z^q(X)$ and a K_1 -chain $\{g_w\} \in G^q(X, 1)$ with support on $|W|$ and $\text{div}(\{g_w\}) = 0$.

Lemma 2.28. ([16, Theorem 4.22]) *In the above notations, suppose that $p+q = d$, $|Z|$ meets $|W|$ properly, the patching systems $\{Z_r^{1,2}\}$ and $\{W_s^{1,2}\}$ satisfy the condition (i) from Lemma 2.24, and that the patching system $\{W_s^{1,2}\}$ satisfies the condition (ii) from Lemma 2.24 with respect to the subvariety $|Z|$. Then we have*

$$\nu_X(m([\{f_z\}] \otimes [W])) = (-1)^{(p+1)q} \left\{ \left(\prod_{z \in Z^{(0)}} f_z^{(\bar{z}, W; x)}(x) \right)_x \right\} \in G^d(X, 1),$$

$$\nu_X(m([Z] \otimes [\{g_w\}])) = (-1)^{pq} \left\{ \left(\prod_{w \in W^{(0)}} g_w^{(Z, \bar{w}; x)}(x) \right)_x \right\} \in G^d(X, 1),$$

where $(\bar{z}, W; x)$ is the intersection index of the subvariety \bar{z} with the cycle W at a point $x \in X^{(d)}$ (the same for $(Z, \bar{w}; x)$).

2.5 Facts on determinant of cohomology and Picard categories

Notions and results of this section are used in Section 4.4.

Let X be a smooth projective variety over a field k . Given two coherent sheaves \mathcal{F} and \mathcal{G} on X , one has a well-defined k^* -torsor $\det R\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X}^L \mathcal{G}) \setminus \{0\}$, where $\det(V^\bullet) = \bigotimes_{i \in \mathbb{Z}} \det^{(-1)^i} H^i(V^\bullet)$ for a bounded complex of vector spaces V^\bullet . We will need to study a behavior of this k^* -torsor with respect to exact sequences of coherent sheaves. With this aim it is more convenient to consider a \mathbb{Z} -graded k^* -torsor

$$\langle \mathcal{F}, \mathcal{G} \rangle = (\text{rk} R\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X}^L \mathcal{G}), \det R\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X}^L \mathcal{G}) \setminus \{0\})$$

and to use the construction of virtual coherent sheaves and virtual vector spaces.

Recall that for any exact category \mathcal{C} , P. Deligne has defined in [11] the category of virtual objects $V\mathcal{C}$ together with a functor $\gamma : \mathcal{C}_{iso} \rightarrow V\mathcal{C}$ that has a certain universal property, where \mathcal{C}_{iso} is the category with the same objects as \mathcal{C} and with morphisms

being all isomorphisms in \mathcal{C} . The category $V\mathcal{C}$ is a *Picard category*: it is non-empty, every morphism in $V\mathcal{C}$ is invertible, there is a functor $+$: $V\mathcal{C} \times V\mathcal{C} \rightarrow V\mathcal{C}$ such that it satisfies some compatible associativity and commutativity constraints and such that for any object L in $V\mathcal{C}$, the functor $(\cdot + L) : V\mathcal{C} \rightarrow V\mathcal{C}$ is an autoequivalence of $V\mathcal{C}$ (see op.cit.)¹. In particular, this implies the existence of a unit object 0 in any Picard category.

Explicitly, the objects of $V\mathcal{C}$ are based loops on the H -space $BQ\mathcal{C}$ and for any two loops γ_1, γ_2 on $BQ\mathcal{C}$, the morphisms in $\text{Hom}_{V\mathcal{C}}(\gamma_1, \gamma_2)$ are the homotopy classes of homotopies from γ_1 to γ_2 . For an object E in \mathcal{C} , the object $\gamma(E)$ in $V\mathcal{C}$ is the canonical based loop on $BQ\mathcal{C}$ associated with E . By $[E]$ denote the class of E in the group $K_0(\mathcal{C})$. Note that $[E] = [E']$ if and only if there is a morphism between $\gamma(E)$ and $\gamma(E')$ in $V\mathcal{C}$. Moreover, an exact sequence in \mathcal{C}

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

defines in a canonical way an isomorphism in $V\mathcal{C}$

$$\gamma(E') + \gamma(E'') \cong \gamma(E).$$

Furthermore, there are canonical isomorphisms $\pi_i(V\mathcal{C}) \cong K_i(\mathcal{C})$ for $i = 0, 1$ and $\pi_i(V\mathcal{C}) = 0$ for $i > 1$.

Let us explain in which sense the functor $\gamma : \mathcal{C}_{iso} \rightarrow V\mathcal{C}$ is universal. Given a Picard category \mathcal{P} , consider the category $\text{Det}(\mathcal{C}, \mathcal{P})$ of *determinant functors*, i.e., a category of pairs (δ, D) , where $\delta : \mathcal{C}_{iso} \rightarrow \mathcal{P}$ is a functor and a D is a functorial isomorphism

$$D(\Sigma) : \delta(E') + \delta(E'') \rightarrow \delta(E)$$

for each exact sequence

$$\Sigma : 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

such that the pair (δ, D) is compatible with zero objects, associativity and commutativity (see op.cit., 4.3). Note that $\text{Det}(\mathcal{C}, \mathcal{P})$ has a natural structure of a Picard category.

For Picard categories \mathcal{P} and \mathcal{Q} , denote by $\text{Fun}^+(\mathcal{P}, \mathcal{Q})$ the category of symmetric monoidal functors $F : \mathcal{P} \rightarrow \mathcal{Q}$. Morphisms in $\text{Fun}^+(\mathcal{P}, \mathcal{Q})$ are monoidal morphisms between monoidal functors. Denote by $0 : \mathcal{P} \rightarrow \mathcal{Q}$ a functor that sends every object of \mathcal{P} to the unit object 0 in \mathcal{Q} and sends every morphism to the identity. Note that $\text{Fun}^+(\mathcal{P}, \mathcal{Q})$ has a natural structure of a Picard category. The universality of $V\mathcal{C}$ is expressed by the following statement (see op.cit., 4.4).

Lemma 2.29. *For any Picard category \mathcal{P} and an exact category \mathcal{C} , the composition with the functor $\gamma : \mathcal{C} \rightarrow V\mathcal{C}$ defines an equivalence of Picard categories*

$$\text{Fun}^+(V\mathcal{C}, \mathcal{P}) \rightarrow \text{Det}(\mathcal{C}, \mathcal{P}).$$

¹In op.cit. this notion is called a *commutative* Picard category; since we do not consider non-commutative Picard categories, we use a shorter terminology.

By universality of $V\mathcal{C}$, the smallest Picard subcategory in $V\mathcal{C}$ containing $\gamma(\mathcal{C}_{iso})$ is equivalent to the whole category $V\mathcal{C}$.

Example 2.30. Let us describe explicitly the category $V\mathcal{M}_k$, where \mathcal{M}_k is the exact category of finite-dimensional vector spaces over a field k (see op.cit., 4.1 and 4.13). Objects of $V\mathcal{M}_k$ are \mathbb{Z} -graded k^* -torsors, i.e, pairs (l, L) , where $l \in \mathbb{Z}$ and L is a k^* -torsor. For any objects (l, L) and (m, M) in $V\mathcal{M}_k$, we have $\text{Hom}_{V\mathcal{M}_k}((l, L), (m, M)) = 0$ if $l \neq m$, and $\text{Hom}_{V\mathcal{M}_k}((l, L), (m, M)) = \text{Hom}_{k^*}(L, M)$ if $l = m$. The functor $(\mathcal{M}_k)_{iso} \rightarrow V\mathcal{M}_k$ sends a vector space V over k to the pair $(\text{rk}_k(V), \det_k(V) \setminus \{0\})$. We have

$$(l, L) + (m, M) = (l + m, L \otimes M)$$

and the commutativity constraint

$$(l, L) + (m, M) \cong (m, M) + (l, L)$$

is given by the formula $u \otimes v \mapsto (-1)^{lm} v \otimes u$, where $u \in L$, $v \in M$.

Given exact categories \mathcal{C} , \mathcal{C}' , \mathcal{E} , and a biexact functor $\mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{E}$, one has a corresponding functor $G : V\mathcal{C} \times V\mathcal{C}' \rightarrow V\mathcal{E}$, which is *distributive* with respect to addition in categories of virtual objects: for a fixed object L in $V\mathcal{C}$ or M in $V\mathcal{C}'$, the functor $G(L, M)$ is a symmetric monoidal functor and the choices of a fixed argument are compatible with each other (see op.cit., 4.11). A distributive functor is an analog for Picard categories of what is a biextension for abelian groups (see Section 3.1).

Now let \mathcal{M}_X , \mathcal{P}_X , and \mathcal{P}'_X denote the categories of coherent sheaves on X , vector bundles on X , and vector bundles E on X such that $H^i(X, E) = 0$ for $i > 0$, respectively. The natural symmetric monoidal functors $V\mathcal{P}_X \rightarrow V\mathcal{M}_X$ and $V\mathcal{P}'_X \rightarrow V\mathcal{P}_X$ are equivalences of categories, see op.cit., 4.12. A choice of corresponding finite resolutions for all objects gives inverse functors to these functors. Thus, there is a distributive functor

$$\langle \cdot, \cdot \rangle : V\mathcal{M}_X \times V\mathcal{M}_X \cong V\mathcal{P}_X \times V\mathcal{P}_X \rightarrow V\mathcal{P}_X \cong V\mathcal{P}'_X \rightarrow V\mathcal{M}_k.$$

There is a canonical isomorphism of the composition $\langle \cdot, \cdot \rangle \circ \gamma : (\mathcal{M}_X)_{iso} \times (\mathcal{M}_X)_{iso} \rightarrow V\mathcal{M}_k$ and the previously defined functor

$$\langle \mathcal{F}, \mathcal{G} \rangle = (\text{rk} R\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X}^L \mathcal{G}), \det R\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X}^L \mathcal{G}) \setminus \{0\})$$

By Lemma 2.29, the last condition defines the distributive functor $\langle \cdot, \cdot \rangle$ up to a unique isomorphism.

In Section 4.4 we will need some more generalities about Picard categories that we describe below. For Picard categories \mathcal{P}_1 and \mathcal{P}_2 , denote by $\text{Distr}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$ the category of distributive functors $G : \mathcal{P}_1 \times \mathcal{P}_2 \rightarrow \mathcal{Q}$. Morphisms between G and G' in $\text{Distr}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$ are morphisms between functors such that for a fixed object L in \mathcal{P}_1 or M in \mathcal{P}_2 , the corresponding morphism of monoidal functors $G(L, M) \rightarrow G'(L, M)$ is monoidal. Denote by $0 : \mathcal{P}_1 \times \mathcal{P}_2 \rightarrow \mathcal{Q}$ a functor that sends every object of $\mathcal{P}_1 \times \mathcal{P}_2$ to the unit object 0 in \mathcal{Q} and sends every morphism to the identity. As above, the category $\text{Distr}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$ has a natural structure of a Picard category.

We will use quotients of Picard categories.

Definition 2.31. Given a functor F in $\text{Fun}^+(\mathcal{P}', \mathcal{P})$, a *quotient* \mathcal{P}/\mathcal{P}' is the following category: objects of \mathcal{P}/\mathcal{P}' are the same as in \mathcal{P} , and morphisms are defined by the formula

$$\text{Hom}_{\mathcal{P}/\mathcal{P}'}(L, M) = \text{colim}_{K \in \text{Ob}(\mathcal{P}')} \text{Hom}_{\mathcal{P}}(L, M + F(K)),$$

i.e., we take the colimit of the functor $\mathcal{P} \rightarrow \text{Sets}$, $K \mapsto \text{Hom}_{\mathcal{P}}(L, M + F(K))$. The composition of morphisms $f \in \text{Hom}_{\mathcal{P}/\mathcal{P}'}(L, M)$ and $g \in \text{Hom}_{\mathcal{P}/\mathcal{P}'}(M, N)$ is defined as follows: represent f and g by elements $\tilde{f} \in \text{Hom}_{\mathcal{P}}(L, M + F(K_1))$ and $\tilde{g} \in \text{Hom}_{\mathcal{P}}(M, N + F(K_2))$, respectively, and take the composition

$$(\tilde{g} + \text{id}_{F(K_1)}) \circ \tilde{f} \in \text{Hom}_{\mathcal{P}}(L, N + F(K_1) + F(K_2)) = \text{Hom}_{\mathcal{P}}(L, N + F(K_1 + K_2)).$$

Remark 2.32.

- (i) Taking a representative $K \in \text{Ob}(\mathcal{P}')$ for each class in $\pi_0(\mathcal{P}')$, we get a decomposition

$$\text{Hom}_{\mathcal{P}/\mathcal{P}'}(L, M) = \coprod_{[K] \in \pi_0(\mathcal{P}')} \text{Hom}_{\mathcal{P}}(L, M + F(K)) / \pi_1(\mathcal{P}'),$$

where $\pi_1(\mathcal{P}')$ acts on $\text{Hom}_{\mathcal{P}}(L, M + F(K))$ via the second summand and the identification $\pi_1(\mathcal{P}') = \text{Hom}_{\mathcal{P}'}(K, K)$.

- (ii) A more natural way to define a quotient of Picard categories would be to consider 2-Picard categories instead of taking quotients of sets of morphisms (or, more generally, to consider homotopy quotients of ∞ -groupoids); the above construction is a truncation to the first two layers in the Postnikov tower of the “right” quotient.

Let us consider a descent property for quotients of Picard categories.

For a symmetric monoidal functor $F : \mathcal{P}' \rightarrow \mathcal{P}$ and a Picard category \mathcal{Q} , consider a category $\text{Fun}_F^+(\mathcal{P}, \mathcal{Q})$, whose objects are pairs (G, Ψ) , where G is an object in $\text{Fun}^+(\mathcal{P}, \mathcal{Q})$ and $\Psi : G \circ F \rightarrow 0$ is an isomorphism in $\text{Fun}^+(\mathcal{P}', \mathcal{Q})$. Morphisms in $\text{Fun}_F^+(\mathcal{P}, \mathcal{Q})$ are defined in a natural way. Note that $\text{Fun}_F^+(\mathcal{P}, \mathcal{Q})$ is a Picard category.

Analogously, for two symmetric monoidal functors of Picard categories $F_1 : \mathcal{P}'_1 \rightarrow \mathcal{P}_1$, $F_2 : \mathcal{P}'_2 \rightarrow \mathcal{P}_2$, and a Picard category \mathcal{Q} , consider a category $\text{Distr}_{(F_1, F_2)}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$, whose objects are triples (G, Ψ_1, Ψ_2) , where G is an object in $\text{Distr}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$, $\Psi_i : G \circ (F_i \times \text{Id}) \rightarrow 0$, $i = 1, 2$, are isomorphisms in $\text{Distr}(\mathcal{P}'_i, \mathcal{P}_{3-i}; \mathcal{Q})$ such that

$$\Psi_1 \circ (\text{Id} \times F_2) = \Psi_2 \circ (F_1 \times \text{Id}) \in \text{Hom}(G \circ (F_1 \times F_2), 0),$$

where morphisms are taken in the category $\text{Distr}(\mathcal{P}'_1, \mathcal{P}'_2; \mathcal{Q})$. Morphisms in $\text{Distr}_{(F_1, F_2)}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$ are defined in a natural way. Note that $\text{Distr}_{(F_1, F_2)}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q})$ is a Picard category.

Lemma 2.33.

- (i) The category \mathcal{P}/\mathcal{P}' inherits a canonical Picard category structure such that the natural functor $P : \mathcal{P} \rightarrow \mathcal{P}/\mathcal{P}'$ is a symmetric monoidal functor.

(ii) *There is an exact sequence of abelian groups*

$$\pi_1(\mathcal{P}') \rightarrow \pi_1(\mathcal{P}) \rightarrow \pi_1(\mathcal{P}/\mathcal{P}') \rightarrow \pi_0(\mathcal{P}) \rightarrow \pi_0(\mathcal{P}') \rightarrow \pi_0(\mathcal{P}/\mathcal{P}') \rightarrow 0.$$

(iii) *The composition of functors with P defines an isomorphism of Picard categories*

$$\mathrm{Fun}^+(\mathcal{P}/\mathcal{P}', \mathcal{Q}) \rightarrow \mathrm{Fun}_F^+(\mathcal{P}, \mathcal{Q}).$$

(iv) *The composition of functors with $P_1 \times P_2$ defines an isomorphism of Picard categories*

$$\mathrm{Distr}(\mathcal{P}_1/\mathcal{P}'_1, \mathcal{P}_2/\mathcal{P}'_2; \mathcal{Q}) \rightarrow \mathrm{Fun}_{(F_1 \times F_2)}^+(\mathcal{P}_1, \mathcal{P}_2; \mathcal{Q}).$$

Proof. (i) Let L, M be objects in \mathcal{P}/\mathcal{P}' ; by definition, the sum $L + M$ in \mathcal{P}/\mathcal{P}' is equal the sum of L and M as objects of the Picard category \mathcal{P} . The commutativity and associativity constraints are also induced by that in the category \mathcal{P} . The check of the needed properties is straightforward.

(ii) Direct checking that uses an explicit description of morphisms given in Remark 2.32(i).

(iii) Let $H : \mathcal{P}/\mathcal{P}' \rightarrow \mathcal{Q}$ be a symmetric monoidal functor and let K be an object in \mathcal{P}' . There is a canonical morphism $F(K) \rightarrow 0$ in \mathcal{P}/\mathcal{P}' given by the morphism $F(K) \rightarrow F(K)$ in \mathcal{P} ; this induces a canonical morphism $(H \circ P \circ F)(K) \rightarrow 0$ in \mathcal{Q} and an isomorphism of symmetric monoidal functors $\Psi : H \circ P \circ F \rightarrow 0$. Thus the assignment $H \mapsto (H \circ P, \Psi)$ defines a functor $\mathrm{Fun}^+(\mathcal{P}/\mathcal{P}', \mathcal{Q}) \rightarrow \mathrm{Fun}_F^+(\mathcal{P}, \mathcal{Q})$.

Let (G, Ψ) be an object in $\mathrm{Fun}_F^+(\mathcal{P}, \mathcal{Q})$. We put $H(L) = G(L)$ for any object L in \mathcal{P}/\mathcal{P}' . Let $f : L \rightarrow M$ be a morphism in \mathcal{P}/\mathcal{P}' represented by a morphism $\tilde{f} : L \rightarrow M + F(K)$ in \mathcal{P} . We put $H(f)$ to be the composition

$$H(L) = G(L) \xrightarrow{G(\tilde{f})} G(M) + (G \circ F)(K) \xrightarrow{id_{G(M)} + \Psi(K)} G(M) = H(M).$$

This defines a symmetric monoidal functor $H : \mathcal{P}/\mathcal{P}' \rightarrow \mathcal{Q}$. The assignment $(G, \Psi) \mapsto H$ defines an inverse functor $\mathrm{Fun}_F^+(\mathcal{P}, \mathcal{Q}) \rightarrow \mathrm{Fun}^+(\mathcal{P}/\mathcal{P}', \mathcal{Q})$ to the functor constructed above.

(iv) The proof is completely analogous to the proof of (iii). \square

For an abelian group A , denote by $\mathrm{Picard}(A)$ a Picard category whose objects are elements of A and whose only morphisms are identities.

Example 2.34.

- (i) Let \mathcal{P} be a Picard category and let $0_{\mathcal{P}}$ be a full subcategory in \mathcal{P} , whose only object is 0; then the natural functor $\mathcal{P}/0_{\mathcal{P}} \rightarrow \mathrm{Picard}(\pi_0(\mathcal{P}))$ is an equivalence of Picard categories.
- (ii) Let \mathcal{A}' be an abelian Serre subcategory in an abelian category \mathcal{A} ; then the natural functor $V\mathcal{A}/V\mathcal{A}' \rightarrow V(\mathcal{A}/\mathcal{A}')$ is an equivalence of Picard categories. This follows immediately from the 5-lemma and Lemma 2.33 (ii).

For an abelian group N , denote by N_{tors} the Picard category of N -torsors. Note that for abelian groups A, B , a biextension of (A, B) by N is the same as a distributive functor $\text{Picard}(A) \times \text{Picard}(B) \rightarrow N_{tors}$. Let $\mathcal{P}_1, \mathcal{P}_2$ be Picard categories, P be a biextension of $(\pi_0(\mathcal{P}_1), \pi_0(\mathcal{P}_2))$ by N . Then P naturally defines a distributive functor $G_P : \mathcal{P}_1 \times \mathcal{P}_2 \rightarrow N_{tors}$.

Combining Lemma 2.33(iv) and Example 2.34(i), we get the next statement.

Lemma 2.35. *A distributive functor $G : \mathcal{P}_1 \times \mathcal{P}_2 \rightarrow N_{tors}$ is isomorphic to the functor G_P for some biextension P of $(\pi_0(\mathcal{P}_1), \pi_0(\mathcal{P}_2))$ by N if and only if $G_*(\pi_0(\mathcal{P}_1) \times \pi_1(\mathcal{P}_2)) = G_*(\pi_1(\mathcal{P}_1) \times \pi_0(\mathcal{P}_2)) = 0 \in N = \pi_1(N_{tors})$.*

3 General facts on biextensions

3.1 Quotient biextension

For all groups below, we write the groups law in the additive manner. For an abelian group A and a natural number $l \in \mathbb{Z}$, let A_l denote the l -torsion in A .

The notion of a biextension was first introduced in [24]; see more details on biextensions in [9] and [28]. Recall that the set of isomorphism classes of biextensions of (A, B) by N is canonically bijective with the group $\text{Ext}_{\mathcal{A}b}^1(A \otimes_{\mathbb{Z}}^L B, N)$ for any abelian groups A, B , and N , where $\mathcal{A}b$ is the category of all abelian groups.

Let us describe one explicit construction of biextensions. As before, let A and B be abelian groups.

Definition 3.1. A subset $T \subset A \times B$ is a *bisubgroup* if for all elements (a, b) , (a, b') , and (a', b) in T , the elements $(a + a', b)$ and $(a, b + b')$ belong to T . For a bisubgroup $T \subset A \times B$ and an abelian group N , a *bilinear map* $\psi : T \rightarrow N$ is a map of sets such that for all elements (a, b) , (a, b') , and (a', b) in T , we have $\psi(a, b) + \psi(a', b) = \psi(a + a', b)$ and $\psi(a, b) + \psi(a, b') = \psi(a, b + b')$.

Suppose that $T \subset A \times B$ is a bisubgroup and $\varphi_A : A \rightarrow A'$, $\varphi_B : B \rightarrow B'$ are surjective group homomorphisms such that $(\varphi_A \times \varphi_B)(T) = A' \times B'$. Consider the set

$$S = T \cap (\text{Ker}(\varphi_A) \times B \cup A \times \text{Ker}(\varphi_B)) \subset A \times B;$$

clearly, S is a bisubgroup in $A \times B$. Given a bilinear map $\psi : S \rightarrow N$, let us define an equivalence relation on $N \times T$ as the transitive closure of the isomorphisms

$$N \times \{(a, b)\} \xrightarrow{\psi(a', b)} N \times \{(a + a', b)\},$$

$$N \times \{(a, b)\} \xrightarrow{\psi(a, b')} N \times \{(a, b + b')\}$$

for all $(a, b) \in T$ and $(a', b), (a, b') \in S$. It is easy to check that the quotient P_ψ of $N \times T$ by this equivalence relation has a natural structure of a biextension of (A', B') by N .

Remark 3.2. Suppose that we are given two bisubgroups $T_2 \subset T_1$ in $A \times B$ satisfying the above conditions; if the restriction of ψ_1 to S_2 is equal to ψ_2 , then the biextensions P_{ψ_1} and P_{ψ_2} are canonically isomorphic.

Remark 3.3. Suppose that we are given two bilinear maps $\psi_1, \psi_2 : S \rightarrow N$. Let $\phi : T \rightarrow N$ be a bilinear map that satisfies $\psi_1 = \phi|_S + \psi_2$; then the multiplication by ϕ defines an isomorphism of the biextensions P_{ψ_2} and P_{ψ_1} of (A', B') by N .

Remark 3.4. There is a natural generalization of the construction mentioned above. Suppose that we are given a biextension Q of (A, B) by N and surjective homomorphisms $\varphi_A : A \rightarrow A'$, $\varphi_B : B \rightarrow B'$. Consider the bisubgroup $S = \text{Ker}(\varphi_A) \times B \cup A \times \text{Ker}(\varphi_B)$ in $A \times B$; a bilinear map (in the natural sense) $\psi : S \rightarrow Q$ is called a *trivialization of the biextension Q over the bisubgroup S* . The choice of the trivialization $\psi : S \rightarrow Q$ canonically defines a biextension P of (A', B') by N such that the biextension $(\varphi_A \times \varphi_B)^* P$ is canonically isomorphic to the biextension Q (one should use the analogous construction to the one described above).

Now let us recall the definition of a Weil pairing associated to a biextension.

Definition 3.5. Consider a biextension P of (A, B) by N and a natural number $l \in \mathbb{N}$, $l \geq 1$; for elements $a \in A_l$, $b \in B_l$, their *Weil pairing* $\phi_l(a, b) \in N_l$ is the obstruction to a commutativity of the diagram

$$\begin{array}{ccc} P_{(a, lb)} & \xrightarrow{\alpha} & P_{(a, b)}^{\wedge l} \\ \uparrow \beta & & \uparrow \gamma \\ P_{(0, 0)} & \xrightarrow{\delta} & P_{(la, b)}, \end{array}$$

where arrows are canonical isomorphisms of N -torsors given by the biextension structure on P : $\phi_l(a, b) = \gamma \circ \delta - \alpha \circ \beta$.

It is easily checked that the Weil pairing defines a bilinear map $\phi_l : A_l \times B_l \rightarrow N_l$.

Remark 3.6. There is an equivalent definition of the Weil pairing: an isomorphism class of a biextension is defined by a morphism $A \otimes_{\mathbb{Z}}^L B[-1] \rightarrow N$ in $D^b(\mathcal{A}b)$, and the corresponding Weil pairing is obtained by taking the composition

$$\oplus_l(A_l \otimes B_l) \rightarrow \text{Tor}_1^{\mathbb{Z}}(A, B) \rightarrow A \otimes_{\mathbb{Z}}^L B[-1] \rightarrow N.$$

Lemma 3.7. Let $\varphi_A : A \rightarrow A'$, $\varphi_B : B \rightarrow B'$, $T \subset A \times B$, and $\psi : S \rightarrow N$ be as above. Consider a pair $(a, b) \in T$ such that $\varphi_A(la) = 0$ and $\varphi_B(lb) = 0$; then we have

$$\phi_l(\varphi_A(a), \varphi_B(b)) = \psi(la, b) - \psi(a, lb),$$

where ϕ_l denotes the Weil pairing associated to the biextension P of (A', B') by N induced by the bilinear map $\psi : S \rightarrow N$.

Proof. By construction the pull-back of the biextension P with respect to the map $T \rightarrow A' \times B'$ is canonically trivial. Furthermore, the pull-back of the diagram that defines the

Weil pairing $\phi_l(\varphi(a), \varphi(b))$ is equal to the following diagram:

$$\begin{array}{ccc} N \times \{(a, lb)\} & \xrightarrow{id_N} & N^{\wedge l} \times \{(a, b)\} \\ \uparrow +\psi(a, lb) & & \uparrow id_N \\ N \times \{(0, 0)\} & \xrightarrow{+\psi(la, b)} & N \times \{(la, b)\}. \end{array}$$

This concludes the proof. \square

3.2 Biextensions and pairings between complexes

We describe a way to construct a biextension starting from a pairing between complexes. Suppose that A^\bullet and B^\bullet are two bounded from above complexes of abelian groups, N is an abelian group, and that we are given a pairing $\phi \in \text{Hom}_{D^-(\mathcal{A}b)}(A^\bullet \otimes_{\mathbb{Z}}^L B^\bullet, N)$. We fix an integer p . Let $H^p(A^\bullet)' \subset H^p(A^\bullet)$ be the annihilator of $H^{-p}(B^\bullet)$ with respect to the induced pairing $\phi : H^p(A^\bullet) \times H^{-p}(B^\bullet) \rightarrow N$. Analogously, we define the subgroup $H^{1-p}(B^\bullet)' \subset H^{1-p}(B^\bullet)$. Let $\tau'_{\leq p} A^\bullet \subset A^\bullet$ be a subcomplex such that $(\tau'_{\leq p} A^\bullet)^n = 0$ if $n > p$, $(\tau'_{\leq p} A^\bullet)^n = A^n$ if $n < p$, and $(\tau'_{\leq p} A^\bullet)^p = \text{Ker}(d_A^p)'$, where $d_A^p : A^p \rightarrow A^{p+1}$ is the differential in the complex A^\bullet and $\text{Ker}(d_A^p)'$ is the group of cocycles that map to $H^p(A^\bullet)'$. Note that the operation $\tau'_{\leq p}$ is well defined for complexes up to quasiisomorphisms, i.e., the class of the complex $\tau'_{\leq p} A^\bullet$ in the derived category $D^-(\mathcal{A}b)$ depends only on the classes of the complexes A^\bullet , B^\bullet and the morphism ϕ . Analogously, we define the subcomplex $\tau'_{\leq (d+1-p)} B^\bullet \subset B^\bullet$. The restriction of ϕ defines an element $\phi' \in \text{Ext}_{D^-(\mathcal{A}b)}^1(\tau'_{\leq p} A^\bullet \otimes_{\mathbb{Z}}^L \tau'_{\leq (1-p)} B^\bullet, N[-1])$. It can be easily checked that ϕ' passes in a unique way through the morphism $\tau'_{\leq p} A^\bullet \otimes_{\mathbb{Z}}^L \tau'_{\leq (1-p)} B^\bullet \rightarrow H^p(A^\bullet)'[-p] \otimes_{\mathbb{Z}}^L H^{1-p}(B^\bullet)'[p-1]$. Thus we get a canonical element

$$\begin{aligned} \phi' &\in \text{Hom}_{D^-(\mathcal{A}b)}(H^p(A^\bullet)'[-p] \otimes_{\mathbb{Z}}^L H^{1-p}(B^\bullet)'[1-p], N) = \\ &= \text{Ext}_{\mathcal{A}b}^1(H^p(A^\bullet)' \otimes_{\mathbb{Z}}^L H^{1-p}(B^\bullet)', N), \end{aligned}$$

i.e., a biextension P_ϕ of $(H^p(A^\bullet)', H^{1-p}(B^\bullet)')$ by N .

We construct this biextension explicitly for the case when ϕ is induced by a true morphism of complexes $\phi : A^\bullet \otimes B^\bullet \rightarrow N$ (this can be always obtained by taking projective resolutions). Let $A = \text{Ker}(d_A^p)'$, $B = \text{Ker}(d_B^{1-p})'$, $T = A \times B$, $\phi_A : \text{Ker}(d_A^p)' \rightarrow H^p(A^\bullet)'$, $\phi_B : \text{Ker}(d_B^{1-p})' \rightarrow H^{1-p}(B^\bullet)'$, $S = \text{Im}(d_A^{p-1}) \times \text{Ker}(d_B^{1-p})' \cup \text{Ker}(d_A^p)' \times \text{Im}(d_B^{-p})$. We define a bilinear map $\psi : S \rightarrow N$ as follows: $\psi(d_A^{p-1}(a^{p-1}), b^{1-p}) = \phi(a^{p-1} \otimes b^{1-p})$ and $\psi(a^p, d_B^{-p}(b^{-p})) = (-1)^p \phi(a^p \otimes b^{-p})$. It is readily seen that this does not depend on the choices of a^{p-1} and b^{-p} , and that $\psi(d_A^{p-1}(a^{p-1}), d_B^{-p}(b^{-p}))$ is well defined (the reason is that ϕ is a morphism of complexes). The application of the construction from Section 3.1 gives a biextension P_ψ of $(H^p(A^\bullet)', H^{1-p}(B^\bullet)')$ by N ; we have $P_\phi = P_\psi$.

Remark 3.8. There is an equivalent construction of the biextension P_ϕ in terms of Picard categories (see Section 2.5). For a two-term complex $C^\bullet = \{C^{-1} \xrightarrow{d} C^0\}$, let $\text{Picard}(C^\bullet)$ be the following Picard category: objects in $\text{Picard}(C^\bullet)$ are elements in the group C^0

and morphisms from $c \in C^0$ to $c' \in C^0$ are elements $\tilde{c} \in C^{-1}$ such that $d\tilde{c} = c' - c$. A morphism of complexes ϕ defines a morphism of complexes

$$\tau_{\geq(p-1)}\tau_{\leq p}A^\bullet \times \tau_{\geq -p}\tau_{\leq(1-p)}A^\bullet \rightarrow N$$

that defines a distributive functor

$$F : \text{Picard}(\tau_{\geq(p-1)}\tau_{\leq p}A^\bullet) \times \text{Picard}(\tau_{\geq -p}\tau_{\leq(1-p)}A^\bullet) \rightarrow BN,$$

where BN is a Picard category with one object 0 and with $\pi_1(BN) = N$. The functor F is defined as follows: for morphisms $\tilde{a} : a \rightarrow a'$ and $\tilde{b} : b \rightarrow b'$, we have $F(\tilde{a}, \tilde{b}) = \phi(\tilde{a} \otimes b') + \phi(a \otimes \tilde{b})$. Consider the restriction of F to the full Picard subcategories that consist of objects whose cohomology classes belong to $H^p(A^\bullet)'$ and $H^{1-p}(B^\bullet)'$, respectively. By Lemma 2.35, we get a biextension P_ϕ of $(H^p(A^\bullet)', H^{1-p}(B^\bullet)')$ by N .

Remark 3.9. In what follows there will be given shifted pairings $p \in \text{Hom}_{D^-(\mathcal{A}b)}(A^\bullet \otimes_{\mathbb{Z}}^L B^\bullet, N[m])$, $m \in \mathbb{Z}$; this gives a biextension of $(H^p(A^\bullet), H^{-m+1-p}(B^\bullet))$ by N .

Example 3.10. Consider a unital DG-algebra A^\bullet over \mathbb{Z} , an abelian group N , and an integer p . Suppose that $A^i = 0$ for $i > d$. Given a homomorphism $H^d(A^\bullet) \rightarrow N$, by $H^p(A^\bullet)' \subset H^p(A^\bullet)$ denote the annihilator of the group $H^{d-p}(A^\bullet)$ with respect to the induced pairing $H^p(A^\bullet) \times H^{d-p}(A^\bullet) \rightarrow N$. Analogously, define the subgroup $H^{d+1-p}(A^\bullet)' \subset H^{d+1-p}(A^\bullet)$. By the above construction, this defines a biextension P of $(H^p(A^\bullet)', H^{d+1-p}(A^\bullet)')$ by N . The associated Weil pairing $\phi_l : H^p(A^\bullet)'_l \times H^{d+1-p}(A^\bullet)'_l \rightarrow N_l$ has the following interpretation as a Massey triple product. For the triple $a \in H^p(A^\bullet)'_l$, $l \in H^0(A^\bullet)$, and $b \in H^{d+1-p}(A^\bullet)'_l$, there is a well defined Massey triple product

$$m_3(a, l, b) \in H^d(A^\bullet)/(a \cdot H^{d-p}(A^\bullet) + H^{p-1}(A^\bullet) \cdot b).$$

By condition, the image $\overline{m}_3(a, l, b) \in N_l$ of $m_3(a, l, b)$ with respect to the map $H^d(A^\bullet) \rightarrow N$ is well defined. By Lemma 3.7, we have $\phi_l(a, b) = \overline{m}_3(a, l, b)$.

Example 3.11. Let Y be a Noetherian scheme of finite dimension d ; then any sheaf of abelian groups \mathcal{F} on Y has no non-trivial cohomology groups with numbers greater than d and there is a natural morphism $R\Gamma(Y, \mathcal{F}) \rightarrow H^d(Y, \mathcal{F})[-d]$ in $D^b(\mathcal{A}b)$. Let \mathcal{F}, \mathcal{G} be sheaves of abelian groups on Y , N be an abelian group, and $p \geq 0$ be an integer. Given a homomorphism $H^d(Y, \mathcal{F} \otimes \mathcal{G}) \rightarrow N$, let $H^p(Y, \mathcal{F})'$ be the annihilator of $H^{d-p}(Y, \mathcal{G})$ with respect to the natural pairing $H^p(Y, \mathcal{F}) \times H^{d-p}(Y, \mathcal{G}) \rightarrow N$. Analogously, we defined the subgroup $H^{d+1-p}(Y, \mathcal{G})' \subset H^{d+1-p}(Y, \mathcal{G})$. Then the multiplication morphism $m \in \text{Hom}_{D^b(\mathcal{A}b)}(R\Gamma(Y, \mathcal{F}) \otimes_{\mathbb{Z}}^L R\Gamma(Y, \mathcal{G}), R\Gamma(Y, \mathcal{F} \otimes \mathcal{G}))$ and the morphism $R\Gamma(Y, \mathcal{F} \otimes \mathcal{G}) \rightarrow N[-d]$ lead to a biextension of $(H^p(Y, \mathcal{F})', H^{d+1-p}(Y, \mathcal{G})')$ by N .

3.3 Poincaré biextension

Consider an example to the construction described above. Let \mathcal{H} be the abelian category of integral mixed Hodge structures. For an object H in \mathcal{H} , by $H_{\mathbb{Z}}$ denote the underlying

finitely generated abelian group, by $W_*H_{\mathbb{Q}}$ denote the increasing weight filtration, and by $F^*H_{\mathbb{C}}$ denote the decreasing Hodge filtration, where $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Recall that if $H_{\mathbb{Z}}$ is torsion-free, then the complex $R\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}(0), H)$ is canonically quasiisomorphic to the complex

$$B(H)^{\bullet} : 0 \rightarrow W_0H_{\mathbb{Z}} \oplus (F^0 \cap W_0)H_{\mathbb{C}} \rightarrow W_0H_{\mathbb{C}} \rightarrow 0,$$

where the differential is given by $d(\gamma, \eta) = \gamma - \eta$ for $\gamma \in W_0H_{\mathbb{Z}}$, $\eta \in (F^0 \cap W_0)H_{\mathbb{C}}$, and $W_0H_{\mathbb{Z}} = H_{\mathbb{Z}} \cap W_0H_{\mathbb{Q}}$ (see [2]). Moreover, for any two integral mixed Hodge structures H and H' with $H_{\mathbb{Z}}$ and $H'_{\mathbb{Z}}$ torsion-free, the natural pairing $R\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}(0), H) \otimes_{\mathbb{Z}}^L R\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}(0), H') \rightarrow R\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}(0), H \otimes H')$ corresponds to the morphism

$$m : B(H)^{\bullet} \otimes_{\mathbb{Z}} B(H')^{\bullet} \rightarrow B(H \otimes H')^{\bullet}$$

given by the formula $m(\varphi \otimes \eta') = \varphi \otimes \eta' \in (H \otimes H')_{\mathbb{C}}$, $m(\gamma \otimes \varphi') = \gamma \otimes \varphi' \in (H \otimes H')_{\mathbb{C}}$, $m(\gamma \otimes \gamma') = \gamma \otimes \gamma' \in W_0(H \otimes H')_{\mathbb{Z}}$, $m(\eta \otimes \eta') = \eta \otimes \eta' \in F^0(H \otimes H')_{\mathbb{C}}$, and $m = 0$ otherwise, where $\gamma \in W_0H_{\mathbb{Z}}$, $\gamma' \in W_0H'_{\mathbb{Z}}$, $\eta \in F^0H_{\mathbb{C}}$, $\eta' \in F^0H'_{\mathbb{C}}$, $\varphi \in H_{\mathbb{C}}$, $\varphi' \in H'_{\mathbb{C}}$.

Let H be a pure Hodge structure of weight -1 , and let $H^{\vee} = \mathrm{Hom}(H, \mathbb{Z}(1))$ be the corresponding internal Hom in \mathcal{H} . By $\langle \cdot, \cdot \rangle : H_{\mathbb{C}} \otimes_{\mathbb{C}} H_{\mathbb{C}}^{\vee} \rightarrow \mathbb{C}^*$ denote the composition of the natural map $H_{\mathbb{C}} \otimes_{\mathbb{C}} H_{\mathbb{C}}^{\vee} \rightarrow \mathbb{C}$ and the exponential map $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}(1) = \mathbb{C}^*$. We put $J(H) = \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Z}(0), H) = H_{\mathbb{C}}/(H_{\mathbb{Z}} + F^0H_{\mathbb{C}})$. This group has a natural structure of a compact complex torus; we call it the *Jacobian of H* . Note that the Jacobian $J(H^{\vee})$ is naturally isomorphic to the dual compact complex torus $J(H)^{\vee}$.

By Section 3.2, the natural pairing

$$\begin{aligned} R\mathrm{Hom}(\mathbb{Z}(0), H) \otimes_{\mathbb{Z}}^L R\mathrm{Hom}(\mathbb{Z}(0), H^{\vee}) &\rightarrow R\mathrm{Hom}(\mathbb{Z}(0), H \otimes H^{\vee}) \rightarrow \\ &\rightarrow R\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Z}(1)) \rightarrow \mathbb{C}^*[-1] \end{aligned}$$

defines a biextension P of $(J(H), J(H^{\vee}))$ by \mathbb{C}^* . The given above explicit description of the pairing m shows that the biextension P is induced by the bilinear map $\psi : S \rightarrow \mathbb{C}^*$ (see Section 3.1), where

$$S = ((H_{\mathbb{Z}} + F^0H_{\mathbb{C}}) \times H_{\mathbb{C}}^{\vee}) \cup (H_{\mathbb{C}} \times (H_{\mathbb{Z}}^{\vee} + F^0H_{\mathbb{C}}^{\vee}))$$

is a bisubgroup in $H_{\mathbb{C}} \times H_{\mathbb{C}}^{\vee}$, $\psi(\gamma + \eta, \varphi^{\vee}) = \langle \gamma, \varphi^{\vee} \rangle$, and $\psi(\varphi, \gamma^{\vee} + \eta^{\vee}) = \langle \varphi, \eta^{\vee} \rangle$ for all $\gamma \in H_{\mathbb{Z}}$, $\gamma^{\vee} \in H_{\mathbb{Z}}^{\vee}$, $\eta \in F^0H_{\mathbb{C}}$, $\eta^{\vee} \in F^0H_{\mathbb{C}}^{\vee}$, $\varphi \in H_{\mathbb{C}}$, $\varphi^{\vee} \in H_{\mathbb{C}}^{\vee}$.

Remark 3.12. The given above explicit construction of the biextension P shows that it coincides with the biextension constructed in [18], Section 3.2. In particular, by Lemma 3.2.5 from op.cit., P is canonically isomorphic to the Poincaré line bundle over $J(H) \times J(H)^{\vee}$.

By Remark 3.12, it makes sense to call P the *Poincaré biextension*.

Remark 3.13. It follows from [18], Section 3.2 that the fiber of the biextension P over a pair $(e, f) \in J(H) \times J(H^{\vee}) = \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Z}(0), H) \times \mathrm{Ext}_{\mathcal{H}}^1(H, \mathbb{Z}(1))$ is canonically bijective with the set of isomorphism classes of integral mixed Hodge structures V whose weight graded quotients are identified with $\mathbb{Z}(0)$, H , $\mathbb{Z}(1)$ and such that $[V/W_{-2}V] = e \in \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Z}(0), H)$, $[W_{-1}V] = f \in \mathrm{Ext}_{\mathcal{H}}^1(H, \mathbb{Z}(1))$.

Remark 3.14. There is a canonical trivialization of the biextension $\log|P|$ of $(J(H), J(H^{\vee}))$ by \mathbb{R} ; this trivialization is given by the bilinear map $H_{\mathbb{C}} \times H_{\mathbb{C}}^{\vee} \rightarrow \mathbb{R}$, $(\varphi, \varphi^{\vee}) \mapsto \log|\langle \varphi, \varphi^{\vee} \rangle|$, where $\varphi = r + \eta$, $r \in H_{\mathbb{R}}$, $\eta \in F^0H$.

4 Biextensions over Chow groups

4.1 Explicit construction

The construction described below was first given in [7]. We use notions and notations from Section 2.2. Let X be a smooth projective variety of dimension d over k . For an integer $p \geq 0$, by $Z^p(X)'$ denote the subgroup in $Z^p(X)$ that consists of cycles Z such that for any K_1 -chain $\{f_\eta\} \in G_{|Z|}^{d-p}(X, 1)$ with $\text{div}(\{f_\eta\}) = 0$, we have $\prod_\eta f_\eta(\overline{\eta} \cap Z) = 1$. Let $CH^p(X)'$ be the image of $Z^p(X)'$ in $CH^p(X)$; by Lemma 2.10, we have $CH^p(X)_{\text{hom}} \subset CH^p(X)'$.

Remark 4.1. If the group k^* has an element of infinite order, then it is easy to see that any cycle $Z \in Z^p(X)'$ is numerically trivial.

Consider integers $p, q \geq 0$ such that $p + q = d + 1$. Let $T \subset Z^p(X)' \times Z^q(X)'$ be the bisubgroup that consists of pairs (Z, W) such that $|Z| \cap |W| = \emptyset$. By the classical moving lemma, we see that the map $T \rightarrow CH^p(X)' \times CH^q(X)'$ is surjective. Let $S \subset T$ be the corresponding bisubgroup as defined in Section 3.1. Define a bilinear map $\psi : S \rightarrow k^*$ as follows. Suppose that Z is rationally trivial; then by Lemma 2.6, there exists an element $\{f_\eta\} \in G_{|W|}^{p-1}(X, 1)$ such that $\text{div}(\{f_\eta\}) = Z$. We put $\psi(Z, W) = \prod_\eta f_\eta(\overline{\eta} \cap W)$. By condition, this element from k^* does not depend on the choice of $\{f_\eta\}$. Similarly, we put $\psi(Z, W) = \prod_\xi g_\xi(Z \cap \overline{\xi})$ if $W = \text{div}(\{g_\xi\})$, $\{g_\xi\} \in G_{|Z|}^{q-1}(X, 1)$. By Lemma 2.8, $\psi(\text{div}(\{f_\eta\}), \text{div}(\{g_\xi\}))$ is well defined. The application of the construction from Section 3.1 yields a biextension P_E of $(CH^p(X)', CH^q(X)')$ by k^* .

The following interpretation of the biextension P_E from Section 4.1 in terms of pairings between higher Chow complexes was suggested to the author by S. Bloch.

We use notations and notions from Section 2.1. As above, let X be a smooth projective variety of dimension d over k and let the integers $p, q \geq 0$ be such that $p + q = d + 1$. There is a push-forward morphism of complexes $\pi_* : Z^{d+1}(X, \bullet) \rightarrow Z^1(\text{Spec}(k), \bullet)$, where $\pi : X \rightarrow \text{Spec}(k)$ is the structure map. Further, there is a morphism of complexes $Z^1(\text{Spec}(k), \bullet) \rightarrow k^*[-1]$ (which is actually a quasiisomorphism). Taking the composition of these morphisms with the multiplication morphism

$$m \in \text{Hom}_{D^-(\mathcal{A}b)}(Z^p(X, \bullet) \otimes_{\mathbb{Z}}^L Z^q(X, \bullet), Z^{d+1}(X, \bullet)),$$

we get an element

$$\phi \in \text{Hom}_{D^-(\mathcal{A}b)}(Z^p(X, \bullet) \otimes_{\mathbb{Z}}^L Z^q(X, \bullet), k^*[-1]).$$

Note that by Lemma 2.1, in notations from Section 3.2, we have $H_0(Z^p(X, \bullet))' = CH^p(X)'$ and $H_0(Z^q(X, \bullet))' = CH^q(X)'$. Hence the construction from Section 3.2 gives a biextension P_{HC} of $(CH^p(X)', CH^q(X)')$ by k^* .

Proposition 4.2. *The biextension P_{HC} is canonically isomorphic to the biextension P_E .*

Proof. Recall that the multiplication morphism m is given by the composition

$$Z^p(X, \bullet) \otimes Z^q(X, \bullet) \xrightarrow{\text{ext}} Z^{d+1}(X \times X, \bullet) \hookrightarrow Z_D^{d+1}(X \times X, \bullet) \xrightarrow{D^*} Z^{d+1}(X, \bullet),$$

where $D \subset X \times X$ is the diagonal. Let $A_0 \subset Z^p(X) \otimes Z^q(X)$ be the subgroup generated by elements $Z \otimes W$ such that $|Z| \cap |W|$, and let

$$A_1 \subset Z^p(X, 1) \otimes Z^q(X, 0) \oplus Z^p(X, 0) \otimes Z^q(X, 1) = (Z^p(X, \bullet) \otimes Z^q(X, \bullet))_1$$

be the subgroup generated by elements $(\alpha \otimes W, Z \otimes \beta)$ such that $\alpha \in Z_{|W|}^p(X, 1)$, $\beta \in Z_{|Z|}^q(X, 1)$; we have $\text{ext}(A_i) \subset Z_D^{d+1}(X \times X, i)$ for $i = 0, 1$. Since all terms of the complex $Z^p(X, \bullet) \otimes Z^q(X, \bullet)$ are free \mathbb{Z} -modules and the groups A_i are direct summands in the groups $(Z^p(X, \bullet) \otimes Z^q(X, \bullet))_i$ for $i = 0, 1$, there exists a true morphism of complexes $\text{ext}' : Z^p(X, \bullet) \otimes Z^q(X, \bullet) \rightarrow Z_D^{p+q}(X \times X, \bullet)$ such that ext' is equivalent to ext in the derived category $D^-(\mathcal{A}b)$ and ext' coincides with ext on A_i , $i = 0, 1$.

On the other hand, for a cycle $Z \in Z^p(X)$ and a K_1 -chain $\{g_\xi\} \in G_{|Z|}^{q-1}(X, 1)$, we have $\beta = \Gamma(\{g_\xi\}) \in Z_{|Z|}^q(X, 1)$ and $(\pi_* \circ D^* \circ \text{ext})(Z \otimes \beta) = \prod_\xi g_\xi(Z \cap \bar{\xi})$ (see Section 2.2).

Therefore the needed statement follows from Remark 3.2 applied to the bisubgroup $T \subset Z^p(X) \times Z^q(X)$ together with the bilinear map ψ from Section 4.1 and the bisubgroup in $Z^p(X) \times Z^q(X)$ together the bilinear map induced by the morphism of complexes $\phi = \pi_* \circ D^* \circ \text{ext}'$ as described in Section 3.2. \square

4.2 Intermediate Jacobians construction

The main result of this section was partially proved in [22] by using the functorial properties of higher Chow groups and the regulator map to Deligne cohomology. We give a more elementary proof that uses only the basic Hodge theory.

We use notions and notations from Sections 3.3 and 2.3. Let X be a complex smooth projective variety of dimension d . Suppose that p and q are natural numbers such that $p + q = d + 1$. Multiplication by $(2\pi i)^p$ induces the isomorphism $J^{2p-1}(X) \rightarrow J(H^{2p-1}(X)(p))$. Since $H^{2p-1}(X)(p)^\vee = H^{2q-1}(X)(q)$, by Section 2.3, there is a Poincaré biextension P_{IJ} of $(J^{2p-1}(X), J^{2q-1}(X))$ by \mathbb{C}^* .

Proposition 4.3. *The pull-back $AJ^*(P_{IJ})$ is canonically isomorphic to the restriction of the biextension P_E constructed in section 4.1 to subgroups $CH^p(X)_{\text{hom}} \subset CH^p(X)'$.*

Proof 1. First, let us give a short proof that uses functorial properties of the regulator map from the higher Chow complex to the Deligne complex.

Let $\mathbb{Z}_D(p)^\bullet = \text{cone}(\mathbb{Z}(p) \oplus F^p \Omega_X^\bullet \rightarrow \Omega_X^\bullet)[-1]$ be the Deligne complex, where all sheaves are considered in the analytic topology. There are a multiplication morphism

$$R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(p)^\bullet) \otimes_{\mathbb{Z}}^L R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(q)^\bullet) \rightarrow R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(p+q)^\bullet)$$

and a push-forward morphism

$$R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(d+1)^\bullet) \rightarrow \mathbb{C}^*[-2d-1]$$

in $D^b(\mathcal{A}b)$. The application of the construction from Section 3.2 to the arising pairing of complexes

$$R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(p)^\bullet) \otimes_{\mathbb{Z}}^L R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(q)^\bullet) \rightarrow \mathbb{C}^*[-2d-1]$$

for $p + q = d + 1$ yields a biextension of $(J^{2p-1}(X), J^{2q-1}(X))$ by \mathbb{C}^* , since $J^{2p-1}(X) \subset H_{an}^{2p}(X(\mathbb{C}), \mathbb{Z}_D(p)^\bullet)'$ and $J^{2q-1}(X) \subset H_{an}^{2q}(X(\mathbb{C}), \mathbb{Z}_D(q)^\bullet)'$.

Furthermore, there is a canonical morphism

$$\epsilon : R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), RH^\bullet(X)(p)) \rightarrow R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(p)^\bullet),$$

where $RH^\bullet(X)$ is the “derived” Hodge structure of X . The morphism ϵ induces isomorphisms on cohomology groups in degrees less or equal to $2p$ (this follows from the explicit formula for the complex $B(H)^\bullet$ given in Section 3.3). Moreover, the morphism ϵ commutes with the multiplication and the push-forward. Note that the multiplication morphism on the left hand side is defined as the composition

$$R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), RH^\bullet(X)(p)) \otimes_{\mathbb{Z}}^L R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), RH^\bullet(X)(q)) \rightarrow$$

$$R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), RH^\bullet(X) \otimes RH^\bullet(X)(p+q)) \rightarrow R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), RH^\bullet(X)(p+q))$$

and the push-forward is defined as the composition

$$\begin{aligned} R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), RH^\bullet(X)(d+1)) &\rightarrow R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), H^{2d}(X)(d+1)[-2d]) = \\ &= R\mathrm{Hom}_{D^b(\mathcal{H})}(\mathbb{Z}(0), \mathbb{Z}(1))[-2d] = \mathbb{C}^*[-2d-1]. \end{aligned}$$

Therefore, the biextension of $(J^{2p-1}(X), J^{2q-1}(X))$ by \mathbb{C}^* defined via the pairing of complexes on the left hand side is canonically isomorphic to the previous one.

The multiplicativity of the spectral sequence

$$\mathrm{Ext}_{\mathcal{H}}^i(\mathbb{Z}(0), H^j(X)(p)) \Rightarrow \mathrm{Hom}_{D^b(\mathcal{H})}^{i+j}(\mathbb{Z}(0), RH^\bullet(X)(p))$$

shows that the last biextension is canonically isomorphic to the Poincaré biextension defined in Section 3.3.

Now consider the cohomological type complex $Z^p(X)^\bullet = Z^p(X, 2p - \bullet)$ built out of the higher Chow complex (see Section 2.1). In [20] it is constructed an explicit morphism

$$\rho : Z^p(X)^\bullet \rightarrow R\Gamma_{an}(X(\mathbb{C}), \mathbb{Z}_D(p)^\bullet)$$

of objects in the derived category $D^-(\mathcal{A}b)$, which is actually the regulator map. It follows from op.cit. that the regulator map ρ commutes with the corresponding multiplication morphisms for higher Chow complexes and Deligne complexes. By Proposition 4.2, this implies immediately the needed result. \square

Question 4.4. Does there exist a morphism of DG-algebras $\rho : A_M^\bullet \rightarrow A_D^\bullet$ such that a DG-algebra A_M^\bullet is quasiisomorphic to the DG-algebra $\bigoplus_{p \geq 0} R\Gamma_{Zar}(X, \mathbb{Z}(p)^\bullet)$, where $\mathbb{Z}(p)^\bullet$

is the Suslin complex, and a DG-algebra A_D^\bullet is quasiisomorphic to the DG-algebra $\bigoplus_{p \geq 0} R\Gamma_{an}(X, \mathbb{Z}_D(p)^\bullet)$? The identification of higher Chow groups with motivic cohomology

$CH^p(X, n) = H_M^{2p-n}(X, \mathbb{Z}(p))$ shows that a positive answer to the question would be implied by the existence of a DG-realization functor from the DG-category of Voevodsky motives (see [4]) to the DG-category associated with integral mixed Hodge structures. It was pointed out to the author by V. Vologodsky that the existence of this DG-realization functor follows immediately from op.cit. Does there exist an explicit construction of the DG-algebras A_M^\bullet , A_D^\bullet , and the morphism ρ ?

Proof 2. Let $\tilde{Z}^p(X)_{\text{hom}}$ be the group that consists of all triples $\tilde{Z} = (Z, \Gamma_Z, \eta_Z)$, where $Z \in Z^p(X)_{\text{hom}}$, Γ_Z is a differentiable singular chain on X such that $\partial\Gamma_Z = Z$, and $\eta_Z \in F_{\log}^p A_{X \setminus |Z|}^{2p-1}$ is such that $\partial_Z([(2\pi i)^p \eta_Z]) = [Z]$. It follows from Section 2.3 that the natural map $\tilde{Z}^p(X)_{\text{hom}} \rightarrow Z^p(X)_{\text{hom}}$ is surjective. There is a homomorphism $\tilde{Z}^p(X) \rightarrow H^{2p-1}(X, \mathbb{C})$ given by the formula $\tilde{Z} \mapsto PD[\Gamma_Z] - [\eta_Z]$; its composition with the natural map $H^{2p-1}(X, \mathbb{C}) \rightarrow J^{2p-1}(X)$ is equal to the Abel–Jacobi map $\tilde{Z}^p(X)_{\text{hom}} \rightarrow J^{2p-1}(X)$ given by the formula $\tilde{Z} \mapsto AJ(Z)$ (see section 2.3).

Let T be a bisubgroup in $\tilde{Z}^p(X)_{\text{hom}} \times \tilde{Z}^q(X)_{\text{hom}}$ that consists of all pairs of triples $(\tilde{Z}, \tilde{W}) = ((Z, \Gamma_Z, \eta_Z), (W, \Gamma_W, \eta_W))$ such that $|Z| \cap |W| = \emptyset$, Γ_Z does not meet $|W|$, and $|Z|$ does not meet Γ_W . Note that for any pair $(\tilde{Z}, \tilde{W}) \in T$, there are well defined classes $PD[\Gamma_Z], [\eta_Z] \in H^{2p-1}(X \setminus |Z|, |W|; \mathbb{C})$ and $PD[\Gamma_W], [\eta_W] \in H^{2q-1}(X \setminus |W|, |Z|; \mathbb{C})$. As before, denote by

$$(\cdot, \cdot) : H^*(X \setminus |Z|, |W|; \mathbb{C}) \times H^{2d-*}(X \setminus |W|, |Z|; \mathbb{C}) \rightarrow \mathbb{C}$$

the natural pairing.

Let $S \subset T$ be a bisubgroup that consists of all pairs of triples (\tilde{Z}, \tilde{W}) such that Z or W is rationally trivial. Let $\psi_1 : S \rightarrow \mathbb{C}^*$ be the pull-back of the bilinear map constructed in Section 4.1 via the natural map $\tilde{Z}^p(X)_{\text{hom}} \times \tilde{Z}^q(X)_{\text{hom}} \rightarrow Z^p(X)_{\text{hom}} \times Z^q(X)_{\text{hom}}$. Let $\psi_2 : S \rightarrow \mathbb{C}^*$ be the pull-back of the bilinear constructed in Section 3.3 via the defined above map $\tilde{Z}^p(X)_{\text{hom}} \times \tilde{Z}^q(X)_{\text{hom}} \rightarrow H^{2p-1}(X, \mathbb{C}) \times H^{2q-1}(X, \mathbb{C})$. The construction from Section 3.1 applied to the maps ψ_1 and ψ_2 gives the biextensions P_E and P_{IJ} , respectively. Define a bilinear map $\phi : T \rightarrow \mathbb{C}^*$ by the formula $\phi(\tilde{Z}, \tilde{W}) = \exp(2\pi i \int_{\Gamma_Z} \eta_W)$. By Remark 3.3, it is enough to show that $\psi_1 = \phi|_S \cdot \psi_2$.

Consider a pair $(\tilde{Z}, \tilde{W}) \in S$. First, suppose that Z is rationally trivial. By linearity and Lemma 2.6, we may assume that $Z = \text{div}(f)$, where $f \in \mathbb{C}^*(Y)$ and $Y \subset X$ is an irreducible subvariety of codimension $p-1$ such that Y meets $|W|$ properly. Let \tilde{Y} be the closure of the graph of the rational function $f : Y \dashrightarrow \mathbb{P}^1$ and let $p : \tilde{Y} \rightarrow X$ be the natural map. Let Γ be a smooth generic path on \mathbb{P}^1 such that $\partial\Gamma = \{0\} - \{\infty\}$ and Γ does not intersect with the finite set $f(p^{-1}(W))$; there is a cohomological class $PD[\Gamma] \in H^1(\mathbb{P}^1 \setminus \{0, \infty\}, f(p^{-1}(|W|)); \mathbb{Z})$. We put $\alpha_Z = (2\pi i)^{-p} [\tilde{Y}]^* (2\pi i PD[\Gamma]) \in H^{2p-1}(X \setminus |Z|, |W|; \mathbb{Z})$. By Lemma 2.17 with $X_1 = \mathbb{P}^1$, $X_2 = X$, $C = \tilde{Y}$, $Z_1 = \{0, \infty\}$, and $W_2 = |W|$, we get $PD[\Gamma_Z] - \alpha_Z \in H^{2p-1}(X, |W|; \mathbb{Z})$ and $\alpha_Z - [\eta_Z] \in F^p H^{2p-1}(X, \mathbb{C})$ (note the element $\alpha_Z - [\eta_Z] \in H^{2p-1}(X, |W|; \mathbb{C})$ does not necessary belong to the subgroup $F^p H^{2p-1}(X, |W|; \mathbb{C})$). Since $PD[\Gamma_Z] - [\eta_Z] = (PD[\Gamma_Z] - \alpha_Z) + (\alpha_Z - [\eta_Z])$, it follows from the explicit construction given in Section 3.3 that

$$\begin{aligned} \psi_2(\tilde{Z}, \tilde{W}) &= \exp(2\pi i (PD[\Gamma_Z] - \alpha_Z, PD[\Gamma_W] - [\eta_W])) = \\ &= \exp(-2\pi i \int_{\Gamma_Z} \eta_W + 2\pi i (\alpha_Z, [\eta_W])). \end{aligned}$$

In addition, combining Lemma 4.5(i) and Lemma 2.17 with $X_1 = X$, $X_2 = \mathbb{P}^1$, $C = \tilde{Y}$, $Z_1 = |W|$, and $W_2 = \{0, \infty\}$, we see that $\exp(2\pi i (\alpha_Z, [\eta_W])) = f(Y \cap W) = \psi_1(\tilde{Z}, \tilde{W})$.

Now suppose that W is rationally trivial. As before, by linearity and Lemma 2.6, we may assume that $W = \text{div}(g)$, where $W \in \mathbb{C}^*(V)$ and $V \subset X$ is an irreducible subvariety

of codimension $p-1$ such that $|Z|$ meets V properly. Let \tilde{V} be the closure of the graph of the rational function $g : V \dashrightarrow \mathbb{P}^1$ and let $p : \tilde{V} \rightarrow X$ be the natural map. Note that the differential form $\frac{dz}{z}$ defines a cohomological class $[\frac{dz}{z}] \in F^1 H^1(\mathbb{P}^1 \setminus \{0, \infty\}, g(p^{-1}(|Z|)); \mathbb{C})$. We put $\alpha_W = (2\pi i)^{-q} [\tilde{V}]^* [\frac{dz}{z}] \in F^q H^{2q-1}(X \setminus |W|, |Z|; \mathbb{C})$. By Lemma 2.17 with $X_1 = \mathbb{P}^1$, $X_2 = X$, $C = \tilde{V}$, $Z_1 = \{0, \infty\}$, and $W_2 = |Z|$, we get $\alpha_W - [\eta_W] \in F^q H^{2q-1}(X, |Z|; \mathbb{Z})$ and $PD[\Gamma_W] - \alpha_W \in H^{2q-1}(X, \mathbb{Z})$ (note that the element $PD[\Gamma_W] - \alpha_W \in H^{2q-1}(X, |Z|; \mathbb{C})$ does not necessary belong to the subgroup $H^{2q-1}(X, |Z|; \mathbb{Z})$). Since $PD[\Gamma_W] - [\eta_W] = (PD[\Gamma_W] - \alpha_W) + (\alpha_W - [\eta_W])$, it follows from the explicit construction given in Section 3.3 that

$$\begin{aligned} \psi_2(\tilde{Z}, \tilde{W}) &= \exp(2\pi i(PD[\Gamma_Z] - [\eta_Z], \alpha_W - [\eta_W])) = \\ &= \exp(2\pi i(PD[\Gamma_Z], \alpha_W) - 2\pi i \int_{\Gamma_Z} \eta_W). \end{aligned}$$

In addition, combining Lemma 4.5(ii) and Lemma 2.17 with $X_1 = X$, $X_2 = \mathbb{P}^1$, $C = \tilde{Y}$, $Z_1 = |Z|$, and $W_2 = \{0, \infty\}$, we see that $\exp(2\pi i(PD[\Gamma_Z], \alpha_W)) = g(Z \cap Y) = \psi_1(\tilde{Z}, \tilde{W})$. This concludes the proof. \square

During the proof of Proposition 4.3 we have used the following simple fact.

Lemma 4.5.

- (i) Let η be a meromorphic 1-form on \mathbb{P}^1 with poles of order at most one (i.e., η is a differential of the third kind), Γ be a smooth generic path on \mathbb{P}^1 such that $\partial\Gamma = \{0\} - \{\infty\}$ and Γ does not contain any pole of η ; if $\text{res}(2\pi i\eta) = \sum_i n_i \{z_i\}$ for some integers n_i , then we have $\exp(2\pi i \int_\Gamma \eta) = \prod_i z_i^{n_i}$.
- (ii) Let z be a coordinate on \mathbb{P}^1 , Γ be a differentiable singular 1-chain on \mathbb{P}^1 that does not intersect with the set $\{0, \infty\}$; if $\partial\Gamma = \sum_i n_i \{z_i\}$, then we have $\exp(\int_\Gamma \frac{dz}{z}) = \prod_i z_i^{n_i}$.

Proof. (i) Let $\log z$ be a branch of logarithm on $\mathbb{P}^1 \setminus \Gamma$, T_ϵ be a tubular neighborhood of Γ with radius ϵ , and let $B_{i,\epsilon}$ be disks around $\{z_i\}$ with radius ϵ ; we put $X_\epsilon = \mathbb{P}^1 \setminus (\cup_i B_{i,\epsilon} \cup T_\epsilon)$. Then we have $0 = \lim_{\epsilon \rightarrow 0} \int_{X_\epsilon} d((\log z)\eta) = \sum_i n_i \log z_i - 2\pi i \int_\Gamma \eta$; this concludes the proof.

(ii) One may assume that Γ does not contain any loop around $\{0\}$ or $\{\infty\}$; hence there exists a smooth path γ on \mathbb{P}^1 such that $\partial\gamma = \{0\} - \{\infty\}$ and γ does not intersect Γ . Let $\log z$ be a branch of logarithm on $\mathbb{P}^1 \setminus \gamma$. Then we have $\int_\Gamma \frac{dz}{z} = \int_\Gamma d \log z = \sum_i n_i \log z_i$; this concludes the proof. \square

Remark 4.6. For a cycle $Z \in Z^p(X)_{\text{hom}}$, consider an exact sequence of integral mixed Hodge structures

$$0 \rightarrow H^{2p-1}(X)(p) \rightarrow H^{2p-1}(X \setminus |Z|)(p) \rightarrow H_{2d-2p}(|Z|) \rightarrow H^{2p}(X)(p).$$

Its restriction to $[Z]_{\mathbb{Z}} = \mathbb{Z}(0) \subset H_{2d-2p}(|Z|)$ defines a short exact sequence

$$0 \rightarrow H^{2p-1}(X)(p) \rightarrow E_Z \rightarrow \mathbb{Z}(0) \rightarrow 0.$$

Analogously, for a cycle $W \in Z^q(X)_{\text{hom}}$, we get an exact sequence

$$0 \rightarrow H^{2q-1}(X)(q) \rightarrow E_W \rightarrow \mathbb{Z}(0) \rightarrow 0$$

and a dual exact sequence

$$0 \rightarrow \mathbb{Z}(1) \rightarrow E_W^\vee \rightarrow H^{2p-1}(X)(p) \rightarrow 0.$$

By Remark 3.13, we see that the fiber $P_{IJ}|_{(AJ(Z), AJ(W))}$ is canonically bijective with the set of isomorphism classes of all integral mixed Hodge structures V whose weight graded quotients are identified with $\mathbb{Z}(0)$, $H^{2p-1}(X)(p)$, $\mathbb{Z}(1)$ and such that $[V/W_{-2}V] = [E_Z] \in \text{Ext}_{\mathcal{H}}^1(\mathbb{Z}(0), H^{2p-1}(X)(p))$, $[W_{-1}V] = [E_W^\vee] \in \text{Ext}_{\mathcal{H}}^1(H^{2p-1}(X)(p), \mathbb{Z}(1))$.

Remark 4.7. Under the assumptions of Corollary 4.32 suppose that $|Z| \cap |W| = \emptyset$. Then there are two canonical trivializations of the fiber of the biextension $\log |P_E|$ over (Z, W) : the first one follows from the construction of P_E and the second one follows from Proposition 4.3 and Remark 3.14. Let $h(Z, W) \in \mathbb{R}$ be the quotient of these two trivializations. Consider closed forms $\eta_Z^r \in F_{\log}^p A_{X \setminus |Z|}^{2p-1}$ and $\eta_W^r \in F_{\log}^q A_{X \setminus |W|}^{2q-1}$ such that $\partial_Z((2\pi i)^p \eta_Z^r) = [Z]$, $\partial_W((2\pi i)^q \eta_W^r) = [W]$, η_Z^r has real periods on $X \setminus |Z|$, and η_W^r has real periods on $X \setminus |W|$ (it is easy to see that such forms always exist). In notations from the proof of Proposition 4.3 the isomorphism $P_E \rightarrow P_{IJ}$ is given by the multiplication by $\exp(2\pi i \int_{\Gamma_Z} \eta_W^r)$. On the other hand, the trivialization from Remark 3.14 is given by the section $\text{Re}(2\pi i (PD[\Gamma_Z] - [\eta_Z^r], PD[\Gamma_W] - [\eta_W^r])) = \text{Re}(2\pi i \int_{\Gamma_Z} (\eta_W^r - \eta_Z^r))$. Therefore, $h(Z, W) = \text{Re}(2\pi i \int_{\Gamma_Z} \eta_W^r)$. This number is called the *archimedean height pairing* of Z and W .

Finally, using Lemma 4.5 we give an explicit analytic proof of Lemma 2.10 for complex varieties.

Lemma 4.8. *Let X be a complex smooth projective variety of dimension d , $W \in Z^q(X)_{\text{hom}}$ be a homologically trivial cycle, $\{Y_i\}$ be a finite collection of irreducible subvarieties of codimension $d - q$ in X , and let $f_i \in \mathbb{C}(Y_i)^*$ be a collection of rational functions such that $\sum_i \text{div}(f_i) = 0$ and for any i , the support $|W|$ meets Y_i properly and $|W| \cap |\text{div}(f_i)| = \emptyset$. Then we have $f(Y \cap W) = \prod_i f_i(Y_i \cap W) = 1$; here for each i , we put $f_i(Y_i \cap W) = \prod_{x \in Y_i \cap |W|} f_i^{(Y_i, W; x)}(x)$, where $(Y_i, W; x)$ is the intersection index of Y_i and W at a point $x \in Y_i \cap |W|$.*

Proof. Since W is homologically trivial, by Section 2.3, there exists a closed differential form $\eta \in F_{\log}^q A_{X \setminus |W|}^{2q-1}$ such that $\partial_W([(2\pi i)^q \eta]) = [W]$, i.e., for a small $(2q - 1)$ -sphere S_j around an irreducible component W_j of W , we have $\int_{S_j} \eta = m_j$, where $W = \sum m_j W_j$.

For each i , let \tilde{Y}_i be the closure of the graph of the rational function $f_i : Y_i \dashrightarrow \mathbb{P}^1$ and let $p_i : \tilde{Y}_i \rightarrow X$ be the natural map. Let γ be a smooth generic path on \mathbb{P}^1 such that $\partial\gamma = \{0\} - \{\infty\}$ and γ does not intersect with the finite set $\Sigma = \cup_i p_i(f_i^{-1}(|W|))$. There is a cohomological class $PD[\gamma] \in H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \Sigma; \mathbb{Z})$; we put $\alpha_i = (2\pi i)^{-(d-q+1)} [\tilde{Y}_i]^* (2\pi i PD[\gamma]) \in H^{2d-2q+1}(X \setminus |\text{div}(f_i)|, |W|; \mathbb{Z})$. Let Γ_i be a differentiable representative for $PD(\alpha_i) \in H_{2q-1}(X \setminus |W|, |\text{div}(f_i)|; \mathbb{Z})$ (see Remark 2.16). In particular, Γ_i is a differentiable singular $(2q - 1)$ -chain on $X \setminus |W|$ with boundary $\text{div}(f_i)$.

By Lemma 4.5 and Lemma 2.17 with $X_1 = \mathbb{P}^1$, $X_2 = X$, $C = \tilde{Y}_i$, $Z_1 = \{0, \infty\}$, and $W_2 = |W|$, we get $f_i(Y_i \cap W) = \exp(2\pi i(\alpha_i, \eta)) = \exp(2\pi i \int_{\Gamma_i} \eta)$. Therefore, $f(Y \cap W) =$

$\exp(2\pi i \int_{\Gamma} \eta)$, where $\Gamma = \sum_i \Gamma_i$. Since $\sum_i \operatorname{div}(f_i) = 0$, the chain Γ on $X \setminus |W|$ has no boundary.

Note that the image of the class α_i under the natural homomorphism

$$H^{2d-2q+1}(X \setminus |\operatorname{div}(f_i)|, |W|; \mathbb{C}) \rightarrow H^{2d-2q+1}(X \setminus |\operatorname{div}(f_i)|, \mathbb{C})$$

belongs to the subgroup $F^{d-q+1} H^{2d-2q+1}(X \setminus |\operatorname{div}(f_i)|, \mathbb{C})$, i.e., for any closed form $\omega \in F^q A_X^{2q-1}$, we have $\int_{\Gamma_i} \omega = 0$. It follows that the chain Γ is homologous to zero on X . Therefore, Γ is homologous on $X \setminus |W|$ to an integral linear combination of small $(2q-1)$ -spheres around irreducible components of the subvariety $|W|$; hence $\int_{\Gamma} \eta \in \mathbb{Z}$ and this concludes the proof. \square

4.3 K -cohomology construction

We use notions and notations from Section 2.4. Let X be a smooth projective variety of dimension d over a field k , and let $p, q \geq 0$ be such that $p + q = d + 1$. There is a product morphism $m : \mathcal{K}_p \otimes \mathcal{K}_q \rightarrow \mathcal{K}_{d+1}$ and a push-forward map $\pi_* : H^d(X, \mathcal{K}_{d+1}) \rightarrow k^*$, where $\pi : X \rightarrow \operatorname{Spec}(k)$ is the structure map. By Example 3.11, for any integer $p \geq 0$, we get a biextension of $(H^p(X, \mathcal{K}_p)', H^q(X, \mathcal{K}_q)')$ by k^* , where $H^p(X, \mathcal{K}_p)' \subset H^p(X, \mathcal{K}_p)$ is the annihilator of the group $H^{q-1}(X, \mathcal{K}_q)$ with respect to the pairing $H^p(X, \mathcal{K}_p) \times H^{q-1}(X, \mathcal{K}_q) \rightarrow k^*$ and the analogous is true for the subgroup $H^q(X, \mathcal{K}_q)' \subset H^q(X, \mathcal{K}_q)$.

Consider a cycle $W \in Z^q(X)$ and a K_1 -chain $\{f_{\eta}\} \in G_{|W|}^{d-q}(X, 1)$ such that $\operatorname{div}(\{f_{\eta}\}) = 0$. By \overline{W} and $\overline{\{f_{\eta}\}}$ denote the classes of W and $\{f_{\eta}\}$ in the corresponding K -cohomology groups. It follows directly from Lemma 2.28 that $\pi_*(m(\overline{\{f_{\eta}\}} \otimes \overline{W})) = \prod_{\eta} f_{\eta}(\overline{\eta} \cap W)$. Hence by Lemma 2.10, the identification $CH^p(X) = H^p(X, \mathcal{K}_p)$ induces the equality $CH^p(X)' = H^p(X, \mathcal{K}_p)'$; thus we get a biextension P_{KC} of $(CH^p(X)', CH^{d+1-1}(X)')$ by k^* .

Proposition 4.9. *The biextension P_{KC} is canonically isomorphic up to the sign $(-1)^{pq}$ to the biextension P_E .*

Proof 1. First, we give a short proof that uses a regulator map from higher Chow groups to K -cohomology. The author is very grateful to the referee who suggested this proof.

Consider the cohomological type higher Chow complex

$$Z^p(X)^{\bullet} = Z^p(X, 2p - \bullet)$$

and the complex of Zariski flabby sheaves $\underline{Z}^p(X)^{\bullet}$ defined by the formula $\underline{Z}^p(X)^{\bullet}(U) = Z^p(U)^{\bullet}$ for an open subset $U \subset X$. Let \mathcal{H}^i be the cohomology sheaves of the complex $\underline{Z}^p(X)^{\bullet}$. It follows from [6] combined with [25] and [29] that $\mathcal{H}^i = 0$ for $i > p$ and that the sheaf \mathcal{H}^p has the following flabby resolution $\underline{Gers}^M(X, p)^{\bullet}$: $\underline{Gers}^M(X, p)^{\bullet}(U) = \underline{Gers}^M(U, p)^{\bullet}$ for an open subset $U \subset X$, where

$$\underline{Gers}^M(U, p)^l = \sum_{\eta \in X^{(l)}} K_{p-l}^M(k(\eta))$$

and K^M denotes the Milnor K -groups. Combining the canonical homomorphism from Milnor K -groups to Quillen K -groups of fields and the fact that the Gersten complex is a resolution of the sheaf \mathcal{K}_p , we get a morphism

$$R\Gamma(X, Z^p(X)^\bullet) \rightarrow R\Gamma(X, \mathcal{K}_p[-p]).$$

Moreover, it follows from [25] and [29] that this morphism commutes with the multiplication morphisms. By Proposition 4.2, we get the needed result. \square

Proof 2. Let us give an explicit proof in the case when the ground field is infinite and perfect. We use adelic complexes (see Section 2.4 and [16]). The multiplicative structure on the adelic complexes defines the morphism of complexes $\phi : \mathbf{A}(X, p)^\bullet \otimes \mathbf{A}(X, q)^\bullet \rightarrow k^*[-d]$, which agrees with the natural morphism $\text{Hom}_{D^b(\mathcal{A}b)}(R\Gamma(X, \mathcal{K}_p) \otimes_{\mathbb{Z}}^L R\Gamma(X, \mathcal{K}_q), k^*[-d])$ that defines the biextension P_{KC} .

By d denote the differential in the adelic complexes $\mathbf{A}(X, n)^\bullet$. In notations from Section 3.2, let $T \subset \text{Ker}(d^p)' \times \text{Ker}(d^q)'$ be the bisubgroup that consists of all pairs $(\sum_i [Z_i], \sum_j [W_j])$ with $[Z_i] \in \mathbf{A}(X, \mathcal{K}_p)^p$, $[W_j] \in \mathbf{A}(X, q)^q$ such that for all i, j , we have:

- (i) the K -adeles $[Z_i] \in \mathbf{A}(X, \mathcal{K}_p)^p$ and $[W_j] \in \mathbf{A}(X, q)^q$ are good cocycles with respect to some patching systems $\{(Z_i)_{r^{1,2}}\}$ and $\{(W_j)_{s^{1,2}}\}$ for cycles $Z_i \in Z^p(X)'$ and $W_j \in Z^q(X)'$ on X , respectively;
- (ii) the patching systems $\{(Z_i)_{r^{1,2}}\}$ and $\{(W_j)_{s^{1,2}}\}$ satisfy the condition (i) from Lemma 2.24 and the patching system $\{(W_j)_{s^{1,2}}\}$ satisfies the condition (ii) from Lemma 2.24 with respect to the subvariety $|Z_i| \subset X$;
- (iii) the support $|Z_i|$ meets the support $|W_j|$ properly.

In particular, for all j , we have $\text{codim}_Z(Z \cap ((W_j)_s^1 \cap (W_j)_s^2)) \geq s + 1$ for all s , $1 \leq s \leq q - 1$, where $Z = |\sum_i Z_i| \in Z^p(X)$.

Combining the classical moving lemma, Lemma 2.24, and Lemma 2.26, we see that the natural map $T \rightarrow H^p(X, \mathcal{K}_p)' \times H^q(X, \mathcal{K}_q)' = CH^p(X)' \times CH^q(X)'$ is surjective.

Suppose that $(\sum_i [Z_i], \sum_j [W_j]) \in T$ and the class of the cycle $Z = \sum_i Z_i$ in $H^p(X, \mathcal{K}_p) = CH^p(X)$ is trivial. By Lemma 2.6 and Corollary 2.7, there exists a K_1 -chain $\{f_\eta\} \in G^{p-1}(X, 1)$ such that $\text{div}(\{f_\eta\}) = Z$, the support $Y = \text{Supp}(\{f_\eta\})$ meets $W = \sum_j W_j \in Z^q(X)$ properly and for all j , we have $\text{codim}_Y(Y \cap ((W_j)_s^1 \cap (W_j)_s^2)) \geq s + 1$ for all s , $1 \leq s \leq q - 1$. It follows that for all j , the patching system $\{(W_j)_{s^{1,2}}\}$ satisfies the condition (ii) from Lemma 2.24 with respect to the subvariety $Y \subset X$. Combining Lemma 2.24, Lemma 2.27, and Lemma 2.28, we see that there exists a K -adele $[\{f_\eta\}] \in \mathbf{A}(X, \mathcal{K}_p)^{p-1}$ such that $d^{p-1}[\{f_\eta\}] = \sum_i [Z_i]$ and $\pi_*(m([\{f_\eta\}] \otimes \sum_j [W_j])) = (-1)^{pq} \prod_\eta f_\eta(\bar{\eta} \cap W)$.

Suppose that $(\sum_i [Z_i], \sum_j [W_j]) \in T$ and the class of the cycle $W = \sum_j W_j$ in $H^q(X, \mathcal{K}_q) = CH^q(X)$ is trivial. By Lemma 2.6, there exists a K_1 -chain $\{g_\xi\} \in G^{q-1}(X, 1)$ such that $\text{div}(\{g_\xi\}) = W$ and the support $Y = \text{Supp}(\{g_\xi\})$ meets $Z = \sum_i Z_i \in Z^p(X)$ properly. Combining Lemma 2.24, Lemma 2.27, and Lemma 2.28, we see that there exists a K -adele $[\{g_\xi\}] \in \mathbf{A}(X, \mathcal{K}_q)^{q-1}$ such that $d^{q-1}[\{g_\xi\}] = \sum_j [W_j]$ and $(-1)^p \pi_*(m([Z] \otimes [\{g_\xi\}])) = (-1)^{pq} \prod_\xi g_\xi(Z \cap \bar{\xi})$.

Therefore by Remark 3.2 applied to $T \subset \text{Ker}(d^p)' \times \text{Ker}(d^q)'$ and ψ induced by $\phi = \pi_* \circ m$, we get the needed result.

Therefore the needed statement follows from Remark 3.2 applied to the bisubgroup $T \subset \text{Ker}(d^p)' \times \text{Ker}(d^q)'$ and the bigger bisubgroup in $\text{Ker}(d^p)' \times \text{Ker}(d^q)'$ together with the bilinear map defined by the morphism of complexes ϕ as shown in Section 3.2. \square

4.4 Determinant of cohomology construction

In Section 2.5 we defined a “determinant of cohomology” distributive functor

$$\langle \cdot, \cdot \rangle : V\mathcal{M}_X \times V\mathcal{M}_X \rightarrow V\mathcal{M}_k.$$

Our goal is to descend this distributive functor to a biextension of Chow groups. The strategy is as follows. First, we define a filtration $C^p V\mathcal{M}_X$ on the category $V\mathcal{M}_X$, which is a kind of a filtration by codimension of support. The successive quotients of this filtration are isomorphic to certain Picard categories \widetilde{CH}_X^p that are related to Chow groups. Then we show a homotopy invariance of the categories $C^p V\mathcal{M}_X$ and construct a specialization map for them. As usual, this allows to define a contravariant structure on the categories $C^p V\mathcal{M}_X$ by using deformation to the normal cone. Next, the exterior product between the categories $C^p V\mathcal{M}_X$ together with the pull-back along diagonal allows one to define for a smooth variety X a collection of distributive functors

$$C^p V\mathcal{M}_X \times C^q V\mathcal{M}_X \rightarrow C^{p+q} V\mathcal{M}_X,$$

compatible with the distributive functor

$$V\mathcal{M}_X \times V\mathcal{M}_X \rightarrow V\mathcal{M}_X$$

defined by the derived tensor product of coherent sheaves. Since all constructions for the categories $C^p V\mathcal{M}_X$ are compatible for different p , we get the analogous constructions for the categories \widetilde{CH}_X^p (contravariancy and a distributive functor). Taking the push-forward functor $C^{d+1} V\mathcal{M}_X \rightarrow V\mathcal{M}_X \rightarrow V\mathcal{M}_k$ for a smooth projective variety X , we get a distributive functor

$$\langle \cdot, \cdot \rangle_{pq} : \widetilde{CH}_X^p \times \widetilde{CH}_X^q \rightarrow V\mathcal{M}_k$$

for $p + q = d + 1$. Finally, applying Lemma 2.35, we get a biextension P_{DC} of $(CH^p(X)', CH^q(X)')$ by k^* . The fiber of this biextension at the classes of algebraic cycles $Z = \sum_i m_i Z_i$ and $W = \sum_j n_j W_j$ is canonically isomorphic to the k^* -torsor

$$\det(\sum_{i,j} m_i n_j R\Gamma(X, \mathcal{O}_{Z_i} \otimes_{\mathcal{O}_X}^L \mathcal{O}_{W_j})) \setminus \{0\}.$$

Also, we prove that the biextension P_{DC} is canonically isomorphic to the biextension P_E constructed in Section 4.1.

Let us follow this plan. For any variety X , and $p \geq 0$ denote by \mathcal{M}_X^p the exact category of sheaves on X whose support codimension is at least p . By definition, put $\mathcal{M}_X^p = \mathcal{M}_X$ for $p < 0$. For $p \geq q$, there are natural functors $V\mathcal{M}_X^p \rightarrow V\mathcal{M}_X$ and $V\mathcal{M}_X^p \rightarrow V\mathcal{M}_X^q$.

Definition 4.10. For $p \geq 0$, let $C^pV\mathcal{M}_X$ be the following Picard category: objects in $C^pV\mathcal{M}_X$ are objects in the category $V\mathcal{M}_X^p$ and morphisms are defined by the formula

$$\mathrm{Hom}_{C^pV\mathcal{M}_X}(L, M) = \mathrm{Im}(\mathrm{Hom}_{V\mathcal{M}_X^{p-1}}(L, M) \rightarrow \mathrm{Hom}_{V\mathcal{M}_X^{p-2}}(L, M))$$

(more precisely, we consider images of objects L and M with respect to the corresponding functors from $V\mathcal{M}_X^p$); a monoidal structure on $C^pV\mathcal{M}_X$ is naturally defined by monoidal structures on the categories $V\mathcal{M}_X^*$.

By definition, we have $C^0V\mathcal{M}_X = V\mathcal{M}_X$. Note that for $p > d + 1$, $C^pV\mathcal{M}_X = 0$ and $C^{d+1}V\mathcal{M}_X$ consists of one object 0 whose automorphisms group is equal to $\mathrm{Im}(K_1(\mathcal{M}_X^d) \rightarrow K_1(\mathcal{M}_X^{d-1}))$ (the last group equals $H^d(X, \mathcal{K}_{d+1})$ provided that X is smooth). For $p \geq q \geq 0$, there are natural functors $C^pV\mathcal{M}_X \rightarrow C^qV\mathcal{M}_X$. It is clear that

$$\begin{aligned}\pi_0(C^pV\mathcal{M}_X) &= \mathrm{Im}(K_0(\mathcal{M}_X^p) \rightarrow K_0(\mathcal{M}_X^{p-1})), \\ \pi_1(C^pV\mathcal{M}_X) &= \mathrm{Im}(K_1(\mathcal{M}_X^{p-1}) \rightarrow K_1(\mathcal{M}_X^{p-2})).\end{aligned}$$

The next definition is the same as the definition given in [12, Section 2].

Definition 4.11. For $p \geq 1$, let \widetilde{CH}_X^p be the following Picard category: objects in \widetilde{CH}_X^p are elements in the group $Z^p(X)$ and morphisms are defined by the formula

$$\mathrm{Hom}_{\widetilde{CH}_X^p}(Z, W) = \{f \in G^{p-1}(X, 1) \mid \mathrm{div}(f) = W - Z\} / \mathrm{Tame},$$

where Tame denotes the K_2 -equivalence on K_1 -chains, i.e., the equivalence defined by the homomorphism

$$\mathrm{Tame} : G^{p-2}(X, 2) \rightarrow G^{p-1}(X, 1);$$

a monoidal structure on \widetilde{CH}_X^p is defined by taking sums of algebraic cycles.

It is clear that for $p \geq 0$, we have

$$\pi_0(\widetilde{CH}_X^p) = CH^p(X),$$

$$\pi_1(\widetilde{CH}_X^p) = H^{p-1}(\mathrm{Gers}(X, p)^\bullet) = \mathrm{Ker}(\mathrm{div}) / \mathrm{Im}(\mathrm{Tame}).$$

Remark 4.12. In notations from Remark 3.8, we have

$$\widetilde{CH}_X^p = \mathrm{Picard}(\tau_{\geq(p-1)}\mathrm{Gers}(X, p)^\bullet).$$

Lemma 4.13. For any $p \geq 0$, there is a canonical equivalence of Picard categories

$$C^pV\mathcal{M}_X / C^{p-1}V\mathcal{M}_X \rightarrow \widetilde{CH}_X^p.$$

Proof. First, let us construct a functor $F^p : C^pV\mathcal{M}_X \rightarrow \widetilde{CH}_X^p$. By Example 2.34 (ii), for any $p \geq 0$, there is an equivalence of Picard categories $H^p : V\mathcal{M}_X^p / V\mathcal{M}_X^{p-1} \rightarrow \bigoplus_{\eta \in X^{(p)}} V\mathcal{M}_{k(\eta)}$. In notation from Section 2.5, this defines a functor $\widetilde{H}^p : V\mathcal{M}_X^p \rightarrow \mathrm{Picard}(Z^p(X))$. We put $F^p(L) = \widetilde{H}^p(L) \in Z^p(X)$ for any object L in $C^pV\mathcal{M}_X$. Let

$f : L \rightarrow M$ be a morphism in $C^p V\mathcal{M}_X$ and let $\tilde{f} : L \rightarrow M$ be a corresponding morphism in the category $V\mathcal{M}_X^{p-1}$ (note that \tilde{f} is not uniquely defined). There are canonical isomorphisms $H^{p-1}(L) \rightarrow 0$, $H^{p-1}(M) \rightarrow 0$, therefore $H^{p-1}(\tilde{f})$ defines a canonical element in the group $\pi_1(\oplus_{\eta \in X^{(p-1)}} V\mathcal{M}_{k(\eta)}) = G^{p-1}(X, 1)$ that we denote again by $H^{p-1}(\tilde{f})$. It is easy to check that $\text{div}(H^{p-1}(\tilde{f})) = F^p(M) - F^p(L)$. We put $F^p(f) = [H^{p-1}(\tilde{f})]$, where brackets denote the class of a K_1 -chain modulo K_2 -equivalence. The exact sequence

$$G^{p-2}(X, 2) \rightarrow K_1(\mathcal{M}_X^{p-1}) \rightarrow K_1(\mathcal{M}_X^{p-2})$$

shows that $[F^{p-1}(\tilde{f})]$ does not depend on the choice of \tilde{f} .

Next, the exact sequences

$$\begin{aligned} K_0(\mathcal{M}_X^{p+1}) &\rightarrow K_0(\mathcal{M}_X^p) \rightarrow Z^p(X), \\ K_1(\mathcal{M}_X^p) &\rightarrow K_1(\mathcal{M}_X^{p-1}) \rightarrow G^{p-1}(X, 1) \end{aligned}$$

show that the composition $C^{p+1}V\mathcal{M}_X \rightarrow C^pV\mathcal{M}_X \rightarrow \widetilde{CH}_X^p$ is canonically trivial. By Lemma 2.33(iii), we get a well-defined functor

$$C^pV\mathcal{M}_X / C^{p-1}V\mathcal{M}_X \rightarrow \widetilde{CH}_X^p.$$

Finally, combining Lemma 2.33(ii) with the explicit description of π_i for all involved categories, we see that the last functor is an isomorphism on the groups π_i , $i = 0, 1$. This gives the needed result. \square

Remark 4.14. Let us construct explicitly an inverse functor $(F^p)^{-1}$ to the equivalence $F^p : C^pV\mathcal{M}_X / C^{p-1}V\mathcal{M}_X \rightarrow \widetilde{CH}_X^p$. For each cycle $Z = \sum_i m_i Z_i$, we put $(F^p)^{-1}(Z) = \sum_i m_i \gamma(\mathcal{O}_{Z_i})$, where we choose an order on the set of summands in the last expression. For a codimension $p-1$ irreducible subvariety $Y \subset X$ and a rational function $f \in k(Y)^*$, let \tilde{Y} be the closure of the graph of the rational function $f : Y \dashrightarrow \mathbb{P}^1$ and let $\pi : \tilde{Y} \rightarrow Y \hookrightarrow X$ be the natural map. Denote by D_0 and D_∞ Cartier divisors of zeroes and poles of f on \tilde{Y} , respectively. Denote by Z_0 and Z_∞ the positive and the negative part of the cycle $\text{div}(f)$ on X , respectively. The exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O}_{\tilde{Y}}(-D_\infty) \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{D_\infty} \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}_{\tilde{Y}}(-D_0) \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{D_0} \rightarrow 0, \end{aligned}$$

and the isomorphism $f : \mathcal{O}_{\tilde{Y}}(-D_\infty) \rightarrow \mathcal{O}_{\tilde{Y}}(-D_0)$ define an element in the set

$$\text{Hom}_{V\mathcal{M}_X^{p-1}}\left(\sum_i (-1)^i \gamma(R^i \pi_* \mathcal{O}_{D_\infty}), \sum_i (-1)^i \gamma(R^i \pi_* \mathcal{O}_{D_0})\right),$$

that in turn defines an element

$$(F^p)^{-1}(f_\eta) \in \text{Hom}_{V\mathcal{M}_X^{p-1}}((F^p)^{-1}(Z_\infty), (F^p)^{-1}(Z_0) + L)$$

up to morphisms in the category $V\mathcal{M}_X^p$, where L is a well-defined object in $V\mathcal{M}_X^{p+1}$. For each morphism in the category \widetilde{CH}_X^p , we choose its representation as the composition of morphisms defined by f for some codimension $p-1$ irreducible subvarieties $Y \subset X$. This allows to extend $(F^p)^{-1}$ to all morphisms in the category \widetilde{CH}_X^p .

Now let us describe some functorial properties of the categories $C^p V\mathcal{M}_X$.

Definition 4.15. For a variety X , let $\{\mathcal{P}_X^p\}$, $p \geq 0$, be the collection of categories $\{V\mathcal{M}_X^p\}$ or $\{C^p \mathcal{M}_X\}$ (the same choice for all varieties), $G_X^{pq} : \mathcal{P}^p \rightarrow \mathcal{P}^q$ be the natural functors, and let S and T be two varieties. A *collection of compatible functors* from $\{\mathcal{P}_S^p\}$ to $\{\mathcal{P}_T^p\}$ is a pair $(\{F^p\}, \{\Psi^{pq}\})$, where $F^p : \mathcal{P}_S^p \rightarrow \mathcal{P}_T^p$, $p \geq 0$, are symmetric monoidal functors and $\Psi^{pq} : G_T^{pq} \circ F^p \rightarrow F^q \circ G_S^{pq}$ are isomorphisms in the category $\text{Fun}^+(\mathcal{P}_S^p, \mathcal{P}_T^q)$ such that the following condition is satisfied: for all $p \geq q \geq r \geq 0$, we have $G_S^{pq}(\Psi^{qr}) \circ G_T^{qr}(\Psi^{pq}) = \Psi^{pr}$.

The proof of the next result is straightforward.

Lemma 4.16. *For varieties S and T , a collection of compatible functors from $\{V\mathcal{M}_S^p\}$ to $\{V\mathcal{M}_T^p\}$ defines in a canonical way a collection of compatible functors from $\{C^p \mathcal{M}_S\}$ to $\{C^p \mathcal{M}_T\}$.*

Example 4.17. Let $f : S \rightarrow T$ be a flat morphism of schemes; then there is a collection of compatible functors $f^* : V\mathcal{M}_T^p \rightarrow V\mathcal{M}_S^p$. By Lemma 4.16, this gives a collection of compatible functors $f^* : C^p \mathcal{M}_T \rightarrow C^p \mathcal{M}_S$.

Now let us prove homotopy invariance of the categories $C^p V\mathcal{M}_X$. Note that there is no homotopy invariance for the categories $V\mathcal{M}_X^p$. We use the following straightforward result on truncations of spectral sequences:

Lemma 4.18.

- (i) *Consider a spectral sequence E_r^{ij} , $r \geq 1$, a natural number s , and an integer $l \in \mathbb{Z}$. Then there is a unique spectral sequence $(t_s^l E)_r^{ij}$, $r \geq s$, such that $(t_s^l E)_s^{ij} = E_s^{ij}$ if $i \geq l$, $(t_s^l E)_s^{ij} = 0$ if $i < l$, and there is a morphism of spectral sequences $(t_s^l E)_r^{ij} \rightarrow E_r^{ij}$, $r \geq s$.*
- (ii) *In the above notations, let s_1, s_2 be two natural numbers; then we have $(t_l^{s_1} E)_r^{ij} = (t_l^{s_2} E)_r^{ij}$ for all $r \geq s = \max\{s_1, s_2\}$ and $i \geq l + s - 1$.*

Lemma 4.19. *Consider a vector bundle $f : N \rightarrow S$; then the natural functors $f^* : C^p V\mathcal{M}_S \rightarrow C^p V\mathcal{M}_N$ are equivalencies of Picard categories.*

Proof. We follow the proof of [14, Lemma 81]. It is enough to show that f^* induces isomorphisms on π_0 and π_1 , i.e., it is enough to show that the natural homomorphisms

$$f^* : \text{Im}(K_n(\mathcal{M}_S^p) \rightarrow K_n(\mathcal{M}_S^{p-1})) \rightarrow \text{Im}(K_n(\mathcal{M}_N^p) \rightarrow K_n(\mathcal{M}_N^{p-1}))$$

are isomorphisms for all $p \geq 0$, $n \geq 0$.

For any variety T , consider Quillen spectral sequence $E_r^{ij}(T)$, $r \geq 1$, that converges to $K_{-i-j}(\mathcal{M}_T)$. In notations from Lemma 4.18, the filtration of abelian categories $\mathcal{M}_T^{p-1} \supset \mathcal{M}_T^p \supset \dots$ defines the spectral sequence that is equal to the shift of the truncated Quillen spectral sequence $(t_1^{p-1} E)_r^{ij}(T)$, $r \geq 1$ and converges to $K_{-i-j}(\mathcal{M}_T^{p-1})$.

By Lemma 4.18, $(t_1^{p-1}E)_r^{ij}(T) = (t_2^{p-1}E)_r^{ij}(T)$ for $i \geq p$. Therefore, there is a spectral sequence $(t_2^{p-1}E)_r^{ij}(T)$, $r \geq 2$ that converges to $\text{Im}(K_{-i-j}(\mathcal{M}_T^p) \rightarrow K_{-i-j}(\mathcal{M}_T^{p-1}))$. Explicitly, we have $(t_2^{p-1}E)_2^{ij}(T) = H^i(\text{Gers}(T, -j)^\bullet)$.

The morphism f defines the morphism of spectral sequences $f^* : (t_2^{p-1}E)_r^{ij}(S) \rightarrow (t_2^{p-1}E)_r^{ij}(N)$, $r \geq 2$. Moreover, the homomorphism f^* is an isomorphism for $r = 2$ (see op.cit.), hence f^* is an isomorphism for all $r \geq 2$. This gives the needed result. \square

Corollary 4.20. *Taking inverse functors to the equivalences $f^* : C^pV\mathcal{M}_V \rightarrow C^pV\mathcal{M}_N$, we get a collection of compatible functors $(f^*)^{-1}$ from $C^pV\mathcal{M}_N$ to $C^pV\mathcal{M}_S$.*

Next we construct a specialization map for the categories $C^pV\mathcal{M}_X$.

Lemma 4.21. *Let $j : D \subset Y$ be a subscheme on a scheme given as a subscheme by the equation $f = 0$, where f is a regular function on Y , and let $U = Y \setminus D$. Then there exists a collection of compatible functors $Sp^p : V\mathcal{M}_U^p \rightarrow V\mathcal{M}_D^p$ such that the composition $V\mathcal{M}_Y \rightarrow V\mathcal{M}_U \xrightarrow{Sp^0} V\mathcal{M}_D$ is canonically isomorphic to j^* in the category of symmetric monoidal functors.*

Proof. The composition $j^* \circ \gamma_Y : (\mathcal{M}_Y)_{iso} \rightarrow V\mathcal{M}_D$ is canonically isomorphic to the functor $\mathcal{F} \mapsto \gamma_D(\mathcal{G}_0) - \gamma_D(\mathcal{G}_{-1})$, where \mathcal{G}_i are cohomology sheaves of the complex $\mathcal{F} \xrightarrow{f} \mathcal{F}$ placed in degrees -1 and 0 . It follows that the composition $j^* \circ j_* \circ \gamma_D : (\mathcal{M}_D)_{iso} \rightarrow V\mathcal{M}_D$ is canonically isomorphic to 0 in the category $\text{Det}(\mathcal{M}_D, V\mathcal{M}_D)$. By Lemma 2.29, this defines a canonical isomorphism $j^* \circ j_* \rightarrow 0$ in the category $\text{Fun}^+(V\mathcal{M}_D, V\mathcal{M}_D)$. Combining Lemma 2.33 (iii) and Example 2.34 (ii), we get the functor $Sp : V\mathcal{M}_U \rightarrow V\mathcal{M}_D$, since there is a natural equivalence of abelian categories $\mathcal{M}_Y/\mathcal{M}_D \rightarrow \mathcal{M}_U$.

More explicitly, the functor $sp : V\mathcal{M}_U \rightarrow V\mathcal{M}_D$ corresponds via Lemma 2.29 to the following determinant functor: $\mathcal{F} \mapsto \gamma_D(j^*\tilde{\mathcal{F}})$, where \mathcal{F} is a coherent sheaf on U and $\tilde{\mathcal{F}}$ is coherent sheaf on Y that restricts to \mathcal{F} (one can easily check that two different collections of choices of $\tilde{\mathcal{F}}$ for all \mathcal{F} define canonically isomorphic determinant functors).

Take a closed subscheme $Z \subset U$ such that each component of Z has codimension at most p in U . Consider the closed subscheme $Z_D = \overline{Z} \times_X D$ in D , where \overline{Z} is the closure of Z in X . We get a functor $Sp_Z : V\mathcal{M}_Z \rightarrow V\mathcal{M}_{Z_D} \rightarrow V\mathcal{M}_D^p$. Given a diagram $Z \xrightarrow{i} Z' \subset U$, we have a canonical isomorphism of symmetric monoidal functors $Sp_{Z'} \circ i_* \rightarrow Sp_Z$. Taking the limits over closed subschemes $Z \subset U$, we get the needed collection of compatible functors $Sp^p : V\mathcal{M}_U^p \rightarrow V\mathcal{M}_D^p$, $p \geq 0$. \square

Corollary 4.22. *Combining Lemma 4.21 and Lemma 4.16, we get a collection of compatible functors $Sp^p : C^pV\mathcal{M}_U \rightarrow C^pV\mathcal{M}_D$ in notations from Lemma 4.21.*

Now we show contravariancy for the categories $C^pV\mathcal{M}_X$. Let $i : S \subset T$ be a regular embedding of varieties; then there is a symmetric monoidal functor

$$i^* : V\mathcal{M}_T \cong V\mathcal{M}'_T \rightarrow V\mathcal{M}_S$$

where \mathcal{M}'_T is a subcategory in \mathcal{M}_T of \mathcal{O}_S -flat coherent sheaves. The composition of $\gamma_T : (\mathcal{M}_T)_{iso} \rightarrow V\mathcal{M}_T$ with i^* is canonically isomorphic to a functor that sends a coherent sheaf \mathcal{F} on T to the object $\sum_i (-1)^i \gamma(\text{Tor}_i^{\mathcal{O}_T}(\mathcal{F}, \mathcal{O}_S))$. By Lemma 2.29, the last condition defines the functor i^* up to a unique isomorphism.

Proposition 4.23. *For each $p \geq 0$, there exists a collection of compatible functors $i_p^* : C^p V\mathcal{M}_T \rightarrow C^p V\mathcal{M}_S$ such that $i_0^* = i^*$.*

Proof. We use deformation to the normal cone (see [13]). Let M be the blow-up of $S \times \{\infty\}$ in $T \times \mathbb{P}^1$ and let $M^0 = M \setminus \mathbb{P}(N)$, where N denotes the normal bundle to S in T . Then we have $M^0|_{\mathbb{A}^1} \cong T \times \mathbb{A}^1$ and $M^0|_{\{\infty\}} = N$, where $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. Combining Corollary 4.20 and Corollary 4.22, we get a compatible collection of functors i_p^* as the composition

$$C^p V\mathcal{M}_T \xrightarrow{pr_T^*} C^p V\mathcal{M}_{T \times \mathbb{A}^1} \xrightarrow{Sp^p} C^p V\mathcal{M}_N \xrightarrow{(pr_S^*)^{-1}} C^p V\mathcal{M}_S,$$

where $pr_T : T \times \mathbb{A}^1 \rightarrow T$, $pr_S : N \rightarrow S$ are natural projections, and $(pr_S^*)^{-1}$ is an inverse to the equivalence $pr_S^* : C^p V\mathcal{M}_S \rightarrow C^p V\mathcal{M}_N$. \square

Remark 4.24. Let $\mathcal{M}_{T,S}^p$ be a full subcategory in \mathcal{M}_T^p whose objects are coherent sheaves \mathcal{F} from \mathcal{M}_T^p whose support meets S properly. Then the composition of the natural functor $(\mathcal{M}_{T,S}^p)_{iso} \rightarrow C^p V\mathcal{M}_T$ with i_p^* is canonically isomorphic to the functor that sends \mathcal{F} in $\mathcal{M}_{T,S}^p$ to the object $\sum_i (-1)^i \gamma(\mathcal{T}or_i^{\mathcal{O}_T}(\mathcal{F}, \mathcal{O}_S))$.

Remark 4.25. Applying Proposition 4.23 to the embedding of the graph of a morphism of varieties $f : X \rightarrow Y$ with smooth Y , we get a collection of compatible functors $f^* : C^p V\mathcal{M}_Y \rightarrow C^p \mathcal{M}_X$. Combining Lemma 4.13 and Lemma 2.33 (iii), we get a collection of symmetric monoidal pull-back functors $f^* : \widetilde{CH}_Y^p \rightarrow \widetilde{CH}_X^p$.

Remark 4.26. The pull-back functors for the categories \widetilde{CH}_X^p are also constructed by J. Franke in [12] by different methods: the categories \widetilde{CH}_X^p are interpreted as equivalent categories to categories of torsors under certain sheaves. Also, a biextension of Chow groups is constructed by the same procedure as below. It is expected that the pull-back functor constructed in op.cit. is canonically isomorphic to the pull-back functor constructed in Remark 4.25. By Proposition 4.31 below, this would imply that the biextension constructed in op.cit. is canonically isomorphic to the biextension P_E .

Remark 4.27. By Lemma 2.21, for a smooth variety X over an infinite perfect field k , there is a canonical equivalence

$$\text{Picard}(\tau_{\geq (p-1)} \tau_{\leq p} \mathbf{A}(X, \mathcal{K}_p)^\bullet) \rightarrow \widetilde{CH}_X^p.$$

Moreover, the Picard categories on the left hand side are canonically contravariant. This gives another way to define a contravariant structure on the Chow categories (note that the Godement resolution for a sheaf \mathcal{K}_p is not enough to do this).

For three varieties R, S, T , a *collection of compatible distributive functors* from $C^p V\mathcal{M}_R \times C^q V\mathcal{M}_S$ to $C^{p+q} V\mathcal{M}_T$, $p, q \geq 0$ is defined analogously to the collection of compatible symmetric tensor functors.

Corollary 4.28. *Let X be a smooth variety. There exists a collection of compatible distributive functors*

$$C^p V\mathcal{M}_X \times C^q V\mathcal{M}_X \rightarrow C^{p+q} V\mathcal{M}_X$$

such that for $p = q = 0$, the composition of the corresponding functor with the functor $(\mathcal{M}_X)_{iso} \times (\mathcal{M}_X)_{iso} \rightarrow V\mathcal{M}_X \times V\mathcal{M}_X$ is canonically isomorphic to the functor $(\mathcal{F}, \mathcal{G}) \mapsto \sum_i (-1)^i \gamma(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$.

Proof. The exterior product of sheaves defines an collection of compatible distributive functors

$$C^p V\mathcal{M}_X \times C^q V\mathcal{M}_X \rightarrow C^{p+q} \mathcal{M}_{X \times X}.$$

Applying Proposition 4.23 to the diagonal embedding $X \subset X \times X$, we get the needed statement. \square

For algebraic cycles $Z = \sum_i m_i Z_i$, $W = \sum_j n_j W_j$, we put

$$\langle Z, W \rangle = \det(\sum_{i,j} m_i n_j R\Gamma(X, \mathcal{O}_{Z_i} \otimes_{\mathcal{O}_X}^L \mathcal{O}_{W_j})) \setminus \{0\}.$$

Proposition 4.29. *Let X be a smooth projective variety of dimension d . Then for $p + q = d + 1$, there is a biextension P_{DC} of $(CH^p(X)', CH^q(X'))$ such that the fiber $P_{DC}|_{\langle Z, W \rangle}$ is canonically isomorphic to the k^* -torsor $\langle Z, W \rangle$.*

Proof. By Lemma 2.33 (iv), the collection of compatible distributive functors from Corollary 4.28 defines a collection of distributive functors

$$\widetilde{CH}_X^p \times \widetilde{CH}_X^q \rightarrow \widetilde{CH}_X^{p+q}.$$

for all $p, q \geq 0$. If $p + q = d + 1$, then we get a distributive functor

$$\langle \cdot, \cdot \rangle_{pq} : \widetilde{CH}_X^p \times \widetilde{CH}_X^q \rightarrow \widetilde{CH}_X^{d+1} = C^{d+1} V\mathcal{M}_X \rightarrow V\mathcal{M}_X \rightarrow V\mathcal{M}_k.$$

Moreover, by Remark 4.14, we have $\langle Z, W \rangle_{pq} = (0, \langle Z, W \rangle)$.

Consider full Picard subcategories $(\widetilde{CH}_X^p)' \subset \widetilde{CH}_X^p$ whose objects are elements $Z \in Z^p(X)'$ and the restriction of the distributive functor $\langle \cdot, \cdot \rangle$ to the product $(\widetilde{CH}_X^p)' \times (\widetilde{CH}_X^q)'$. Applying Lemma 2.35, we get the next result. \square

Finally, let us compare the biextension P_{DC} with the biextension P_E . With this aim we will need the following result.

Lemma 4.30. *Let X be a smooth projective variety of dimension d , $p + q = d + 1$, let $Y \subset X$ be a codimension $p - 1$ irreducible subvariety, $f \in k(Y)^*$, and let $W \subset X$ be a codimension q irreducible subvariety. Suppose that Y meets W properly and that $|\text{div}(f)| \cap W = \emptyset$. Then f defines a isomorphism $0 \rightarrow \text{div}(f)$ in \widetilde{CH}_X^p , hence a morphism $\sigma(f) : k^* = \langle 0, W \rangle \rightarrow \langle \text{div}(f), W \rangle = k^*$ such that $\sigma(f) = f(Y \cap W)$.*

Proof. By Remark 4.14 and in its notations, the action of f in question is induced by the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\widetilde{Y}}(-D_\infty) \rightarrow \mathcal{O}_{\widetilde{Y}} \rightarrow \mathcal{O}_{D_\infty} \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_{\widetilde{Y}}(-D_0) \rightarrow \mathcal{O}_{\widetilde{Y}} \rightarrow \mathcal{O}_{D_0} \rightarrow 0, \end{aligned}$$

and the isomorphism $f : \mathcal{O}_{\widetilde{Y}}(-D_\infty) \rightarrow \mathcal{O}_{\widetilde{Y}}(-D_0)$. Moreover, $\pi : \widetilde{Y} \rightarrow Y$ is an isomorphism outside of $D_0 \cup D_\infty$, so $\sigma(f)$ is induced by the automorphism of the complex $\mathcal{O}_Y \otimes_{\mathcal{O}_X}^L \mathcal{O}_W$ given by multiplication on f . Applying the functor $\det R\Gamma(-)$, we get $f(Y \cap W)$. \square

Proposition 4.31. *The biextension P_{DC} is canonically isomorphic to the biextension P_E .*

Proof. Let $T \subset Z^p(X)' \times Z^q(X)'$ be the bisubgroup that consists of pair (Z, W) such that $|Z| \cap |W| = \emptyset$. Suppose that $(Z, W), (Z', W)$ are in T and that Z is rationally equivalent to Z' . By Lemma 2.6, we may suppose that there is a K_1 -chain $\{f_\eta\} \in G_{|W|}^{p-1}(X, 1)$ such that $\text{div}(\{f_\eta\}) = Z' - Z$. By Lemma 4.30, we immediately get the needed result. \square

Combining Proposition 4.31, Proposition 4.3, and Remark 4.6, we obtain the following statement.

Corollary 4.32. *Suppose that X is a complex smooth projective variety of dimension d , $Z \in Z^p(X)_{\text{hom}}$, $W \in Z^q(X)_{\text{hom}}$, $p + q = d + 1$; then there is a canonical isomorphism of the following \mathbb{C}^* -torsors:*

- $\det R\Gamma(X, \mathcal{O}_Z \otimes_{\mathcal{O}_X}^L \mathcal{O}_W) \setminus \{0\}$;
- *the set of isomorphism classes of all integral mixed Hodge structures V whose weight graded quotients are identified with $\mathbb{Z}(0)$, $H^{2p-1}(X)(p)$, $\mathbb{Z}(1)$ and such that $[V/W_{-2}V] = [E_Z] \in \text{Ext}_{\mathcal{H}}^1(\mathbb{Z}(0), H^{2p-1}(X)(p))$, $[W_{-1}V] = [E_W^\vee] \in \text{Ext}_{\mathcal{H}}^1(H^{2p-1}(X)(p), \mathbb{Z}(1))$.*

Question 4.33. Does there exist a direct proof of Corollary 4.32?

The present approach is a higher-dimensional generalization for the description of the Poincaré biextension on a smooth projective curve C in terms of determinant of cohomology of invertible sheaves, suggested by P. Deligne in [11]. Let us briefly recall this construction for the biextension \mathcal{P} of $(\text{Pic}^0(C), \text{Pic}^0(C))$ by k^* . For any degree zero invertible sheaves \mathcal{L} and \mathcal{M} on C there is an equality

$$\mathcal{P}_{([\mathcal{L}], [\mathcal{M}])} = \langle \mathcal{L} - \mathcal{O}_C, \mathcal{M} - \mathcal{O}_C \rangle = \det R\Gamma(C, \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{M} - \mathcal{L} - \mathcal{M} + \mathcal{O}_C) \setminus \{0\},$$

where $[\cdot]$ denotes the isomorphism class of an invertible sheaf. Since $\chi(C, \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{M}) = \chi(C, \mathcal{L}) = \chi(C, \mathcal{M}) = 1 - g$, this k^* -torsor is well defined on $\text{Pic}^0(C) \times \text{Pic}^0(C)$. Consider a homomorphism $p : \text{Pic}^0(C) \times \text{Div}^0(C) \rightarrow \text{Pic}^0(C) \times \text{Pic}^0(C)$ given by the formula $([\mathcal{L}], E) \mapsto ([\mathcal{L}], [\mathcal{O}_C(E)])$. Then there is an isomorphism of k^* -torsors $\varphi : p^*\mathcal{P} \cong \mathcal{P}'$, where \mathcal{P}' is a biextension of $(\text{Pic}^0(C), \text{Div}^0(C))$ by k^* given by the formula $\mathcal{P}'_{([\mathcal{L}], E)} = (\bigotimes_{x \in C} \mathcal{L}|_x^{\otimes \text{ord}_x(E)}) \setminus \{0\}$. Thus φ induces a biextension structure on $p^*\mathcal{P}$; it turns out that this structure descends to \mathcal{P} . Moreover, in op.cit. it is proved that the biextension \mathcal{P} is canonically isomorphic to the Poincaré line bundle on $\text{Pic}^0(X) \times \text{Pic}^0(X)$ without the zero section.

Claim 4.34. *In the above notations, the biextensions \mathcal{P} and P_{DC} are canonically isomorphic.*

Proof. Consider a map $\pi : \text{Div}^0(C) \times \text{Div}^0(C) \rightarrow \text{Pic}^0(C) \times \text{Pic}^0(C)$ given by the formula $(D, E) \mapsto ([\mathcal{O}_C(D), \mathcal{O}_C(E)])$. There are canonical isomorphisms of k^* -torsors

$$\begin{aligned} \pi^* \mathcal{P} &\cong \left(\bigotimes_{x \in C} \mathcal{O}_C(D)|_x^{\otimes \text{ord}_x(E)} \right) \setminus \{0\} \cong \left(\bigotimes_{x \in C} \mathcal{O}_C(x)|_x^{\otimes (\text{ord}_x(D) + \text{ord}_x(E))} \right) \setminus \{0\} \cong \\ &\cong \left\langle \sum_{x \in C} \text{ord}_x(D) \mathcal{O}_x, \sum_{y \in C} \text{ord}_y(E) \mathcal{O}_y \right\rangle. \end{aligned}$$

Besides, these isomorphisms commute with the biextension structure. It remains to note that the rational equivalence on the first argument also commutes with the above isomorphisms (this is a particular case of Lemma 4.30). \square

Remark 4.35. For a smooth projective curve, the biextension P_{KC} from Section 4.3 corresponds to a pairing between complexes given by a certain product of Hilbert tame symbols. Combining Proposition 4.9, Proposition 4.31, and Claim 4.34, we see that there is a connection between the tame symbol and the Poincaré biextension given in terms of determinant of cohomology of invertible sheaves. In [15] this connection is explained explicitly in terms of the central extension of ideles on a smooth projective curves constructed by Arbarello, de Concini, and Kac.

4.5 Consequences on the Weil pairing

Let X be a smooth projective variety of dimension d over a field k . As above, suppose that $p, q \geq 0$ are such that $p + q = d + 1$. Consider cycles $Z \in Z^p(X)'$ and $W \in Z^q(X)'$ (see Section 4.1) such that $lZ = \text{div}(\{f_\eta\})$ and $lW = \text{div}(\{g_\xi\})$ for an integer $l \in \mathbb{Z}$ and K_1 -chains $\{f_\eta\} \in G_{|W|}^{p-1}(X, 1)$, $\{g_\xi\} \in G_{|Z|}^{q-1}(X, 1)$ (see Section 2.2). By $[Z]$ and $[W]$ denote the classes of the cycles Z and W in the groups $CH^p(X) = H^p(X, \mathcal{K}_p)$ and $CH^q(X) = H^q(X, \mathcal{K}_q)$, respectively. As explained in Example 3.10, the Massey triple product in K -cohomology

$$m_3([Z], l, [W]) \in H^d(X, \mathcal{K}_{d+1}) / ([Z] \cdot H^{q-1}(X, \mathcal{K}_q) + H^{p-1}(X, \mathcal{K}_p) \cdot [W])$$

has a well defined push-forward $\overline{m}_3([Z], l, [W]) \in \mu_l$ with respect to the map $\pi_* : H^d(X, \mathcal{K}_{d+1}) \rightarrow k^*$, where $\pi : X \rightarrow \text{Spec}(k)$ is the structure morphism.

Combining Lemma 3.7, Example 3.10, Proposition 4.9, and Proposition 4.3, we get the following statements.

Corollary 4.36.

(i) *In the above notations, we have*

$$\overline{m}_3(\alpha, l, \beta)^{(-1)^{pq}} = \prod_{\eta} f_{\eta}(\overline{\eta} \cap W) \cdot \prod_{\xi} g_{\xi}^{-1}(Z \cap \overline{\xi}) = \phi_l([Z], [W]),$$

where ϕ_l is the Weil pairing associated with the biextension P_E from Section 4.1.

(ii) If $k = \mathbb{C}$ and $Z \in Z^p(X)_{\text{hom}}$, $W \in Z^q(X)_{\text{hom}}$, then we have

$$\phi_l([Z], [W]) = \phi_l^{an}(AJ(Z), AJ(W)),$$

where ϕ_l^{an} is the Weil pairing between the l -torsion subgroups in the dual complex tori $J^{2p-1}(X)$ and $J^{2q-1}(X)$.

This is a generalization of the classical Weil's formula for divisors on a curve. The first part of Corollary 4.36 was proved in [16] without considering biextensions.

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